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LUCIEN AUGUSTUS WAIT . . . General Editor

(SENIOR PROFESSOR OF MATHEMATICS IN CORNELL UNIVERSITY)

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(*Senior Professor of Mathematics in Cornell University,*)
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AN ELEMENTARY COURSE

IN THE

INTEGRAL CALCULUS

BY

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NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

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MURRAY'S INTEG. CALC.
W. P. 9

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PREFACE

THIS book has been written primarily for use at Cornell University and similar institutions. In this university the classes in calculus are composed mainly of students in engineering for whom an elementary course in the Integral Calculus is prescribed for the third term of the first year. Their purpose in taking the course is to acquire facility in performing easy integrations and the power of making the simple applications which arise in practical work. While the requirements of this special class of students have been kept in mind, care has also been taken to make the book suitable for any one beginning the study of this branch of mathematics. The volume contains little more than can be mastered by a student of average ability in a few months, and an effort has been made to present the subject-matter, which is of an elementary character, in a simple manner.

The object of the first two chapters is to give the student a clear idea of what the Integral Calculus is, and of the uses to which it may be applied. As this introduction is somewhat longer than is usual in elementary works on the calculus, some teachers may, perhaps, prefer to postpone the reading of several of the articles until the student has had a certain amount of practice in the processes of integration. It is believed, however, that a careful study of Chapters I., II., will arouse the student's interest and quicken his understanding of the subject. There may be some difference of opinion also as to whether the beginner should be introduced to the subject through Chapter I. or through Chapter II. The decision of this question will depend upon the point of view of the individual teacher. So far as the remaining portion of the book is concerned it is a matter of indifference which of these chapters is taken first; and, with slight modifications, they can be interchanged. In Chapter III. the fundamental rules and methods of integration are explained. Since it has been deemed advisable to introduce practical applications as early as possible, Chapter IV. is devoted to the determination of plane areas and of volumes of solids of revolution. The subject of Integral Curves, which is of especial importance to the engineer, is treated in Chapter XII.

Many of the examples are original. Others, especially some of those given in the practical applications, by reason of their nature and importance, are common to all elementary courses on calculus. In several instances, examples of particular interest have been drawn from other works.

A list of lessons suggested for a short course of eleven or twelve weeks is given on page viii. This list has been arranged so that four lessons and a review will be a week's work.

It is hardly possible to name all the sources from which the writer of an elementary work may have obtained suggestions and ideas. I am especially conscious, however, of my indebtedness to the treatises of De Morgan, Williamson, Edwards, Stegemann and Kiepert, and Lamb.

To my colleagues in the department of mathematics at Cornell University, I am under obligations for many valuable criticisms and suggestions. Both the arrangement and the contents have been influenced in a large measure by our conferences and discussions. As originally projected, the volume was to have been written in collaboration with Dr. Hutchinson, but circumstances prevented the carrying-out of this plan.

Chapters V., VI., in part, and Articles 28, 73, in their entirety, have been contributed by him. My colleagues have aided me also in correcting the proofs.

From Professor I. P. Church of the College of Civil Engineering and Professor W. F. Durand of the Sibley College of Mechanical Engineering, I have received valuable suggestions for making the book useful to engineering students. Professor Durand kindly placed at my disposal, with other notes, his article on "Integral Curves" in the Sibley Journal of Engineering, Vol. XI., No. 4; and Chapter XII. is, with slight changes, a reproduction of that article. I take this opportunity of thanking Mr. A. T. Bruegel, Instructor in the kinematics of machinery, and Mr. Murray Macneill, Fellow in mathematics in this university, the former for the interest and care taken by him in drawing the figures, the latter for his assistance in verifying examples and reading proof sheets.

D. A. MURRAY.

CORNELL UNIVERSITY.

LIST OF LESSONS SUGGESTED FOR A SHORT COURSE

[Reviews to follow Every Fourth Lesson]

1. Arts. 1, 2, 3.
2. Arts. 4, 5, 6, 7.
3. Arts. 8, 9, 10, 11.
4. Arts. 12, 17, 18, to Ex. 9.
5. Arts. 13, 18, Exs. 10–12, 19.
6. Arts. 14, 20.
7. Arts. 15, 16, 21.
8. Arts. 22, 23, Exs. 1–23, odd examples.
9. Art. 23, Exs. 24–32, 24, 25, Exs. 1–8, page 55.
10. Pages 56, 57, even or odd examples, Exs. 9–41, Exs. 42–47.
11. Arts. 26, 27, Exs. 1, 2, page 68.
12. Arts. 28, 29, Exs. 3, 4, page 68.
13. Exs. 5–14, page 68.
14. Art. 30.
15. Arts. 31, 32.
16. Page 76, Exs. 1–15.
17. Pages 76, 77, Exs. 16–26.
18. Arts. 33, 34.
19. Arts. 35, 36. Selected examples.
20. Arts. 37, 38, 39, 40.
21. Arts. 42, 43.
22. Art. 45.
23. Selected examples, pages 98, 99.
24. Arts. 46, 47, 48 (*a*).
25. Arts. 48 (*d*), 49, 50 (*a*), (*c*).
26. Art. 51 (*a*).
27. Arts. 52, 53.
28. Arts. 58, 59, 60.
29. Arts. 61, 62.
30. Arts. 64, 65, 66.
31. Arts. 67, 68, 69.
32. Arts. 63, 70. Selected examples, pages 164, 165.
33. Art. 71. Selected examples, page 165.
34. Art. 72. Selected examples, page 166.
35. Art. 73.
36. Art. 74. Selected examples, pages 164, 165.
37. Art. 75, Ex. 9, page 164.
38. Art. 76. Selected examples, Art. 77, Ex. 1.
39. Selected examples, pages 162–166.
40. Arts. 78, 79. Selected examples.
41. Art. 80. Selected examples.
42. Arts. 81, 82, 84, 85.
43. Arts. 86, 87, 88.
44. Arts. 89, 91, 92, 94, 95.

CONTENTS

CHAPTER I

Integration a Process of Summation

Art. Page
1. Uses of the integral calculus. Definition and sign of integration . 1
2. Illustrations of the summation of infinitesimals 2
3. Geometrical principle 7
4. Fundamental theorem. Definite integral 8
5. Supplement to Art. 3 13
6. Geometrical representation of an integral 14
7. Properties of definite integrals 15

CHAPTER II

Integration the Inverse of Differentiation

8. Integration the inverse of differentiation 18
9. Indefinite integral. Constant of integration 22
10. Geometrical meaning of the arbitrary constant of integration . 23
11. Relation between the indefinite and the definite integral . . 24
12. Examples that involve anti-differentials 25
13. Another derivation of the integration formula for an area . . 27
14. A new meaning of y in the curve whose equation is $y = f(x)$. Derived curves 29
15. Integral curves 33
16. Summary 35

CHAPTER III

Fundamental Rules and Methods of Integration

18. Fundamental integrals 36
19. Two universal formulæ of integration 39
20. Integration aided by a change of the independent variable . . 41
21. Integration by parts 44
22. Additional standard forms 47
23. Derivation of the additional standard forms 48

ART. PAGE
24. Integration of a total differential 52
25. Summary 54
Examples on Chapter III. 55

CHAPTER IV

GEOMETRICAL APPLICATIONS OF THE CALCULUS

26. Applications of the calculus 58
27. Areas of curves, rectangular coördinates 58
28. Precautions to be taken in finding areas by integration . . . 63
29. Precautions to be taken in evaluating definite integrals . . . 67
30. Volumes of solids of revolution 69
31. On the graphical representation of a definite integral . . . 73
32. Derivation of the equations of certain curves 74
Examples on Chapter IV. 76

CHAPTER V

RATIONAL FRACTIONS

34. Case I. 79
35. Case II. 81
36. Case III. 82

CHAPTER VI

IRRATIONAL FUNCTIONS

38. The reciprocal substitution 84
39. Trigonometric substitutions 85
40. Expressions containing fractional powers of $a + bx$ only . . 85
41. Functions of the form $f\left\{x^2, (a + bx^2)^{\frac{m}{n}}\right\} \cdot x\,dx$, in which m, n, are integers 87
42. Functions of the form $F(x, \sqrt{x^2 + ax + b})\,dx$, $F(u, v)$ being a rational function of u, v 88
43. Functions of the form $f(x, \sqrt{-x^2 + ax + b})\,dx$, $f(u, v)$ being a rational function of u, v 89
44. Particular functions involving $\sqrt{ax^2 + bx + c}$ 91
45. Integration of $x^m(a + bx^n)^p\,dx$: (a) by the method of undetermined coefficients; (b) by means of reduction formulæ . . . 93
Examples on Chapter VI. 98

CHAPTER VII

INTEGRATION OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

ART. PAGE
46. $\int \sin^n x\,dx, \int \cos^n x\,dx$, n being an integer 100
47. Algebraic transformations 103
48. $\int \sec^n x\,dx, \int \operatorname{cosec}^n x\,dx$ 103
49. $\int \tan^n x\,dx, \int \cot^n x\,dx$ 106
50. $\int \sin^m x \cos^n x\,dx$ 107
51. Integration of $\sin^m x \cos^n dx$: (a) by the method of undetermined coefficients; (b) by means of reduction formulæ . . . 109
52. $\int \tan^m x \sec^n x\,dx, \int \cot^m x \operatorname{cosec}^n x\,dx$ 113
53. Use of multiple angles 114
54. $\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ 115
55. $\int \frac{dx}{a + b\cos x}, \int \frac{dx}{a + b \sin x}$ 115
56. $\int e^{ax} \sin nx\,dx, \int e^{ax} \cos nx\,dx$ 117
57. $\int \sin mx \cos nx\,dx, \int \cos mx \cos nx\,dx, \int \sin mx \sin nx\,dx$ . . 118

CHAPTER VIII

SUCCESSIVE INTEGRATION. MULTIPLE INTEGRALS

58. Successive integration 119
59. Successive integration with respect to a single independent variable 119
60. Successive integration with respect to two or more independent variables 123
61. Application of successive integration to the measurement of areas: rectangular coördinates 126
62. Application of successive integration to the measurement of volumes: rectangular coördinates 128
63. Further application of successive integration to the measurement of volumes: polar coördinates 131

CHAPTER IX

FURTHER GEOMETRICAL APPLICATIONS. MEAN VALUES

65. Derivation of the equations of curves in polar coördinates . . . 134
66. Areas of curves when polar coördinates are used: by single integration 135

ART. PAGE
67. Areas of curves when polar coördinates are used: by double integration 139
68. Areas in Cartesian coördinates with oblique axes 140
69. Integration after a change of variable. Integration when the variables are expressed in terms of another variable 141
70. Measurement of the volumes of solids by means of infinitely thin cross-sections 142
71. Lengths of curves: rectangular coördinates 144
72. Lengths of curves: polar coördinates 147
73. The intrinsic equation of a curve 149
74. Areas of surfaces of solids of revolution 152
75. Areas of surfaces whose equations have the form $z = f(x, y)$. 156
76. Mean values 160
77. A more general definition of mean value 163
Examples on Chapter IX. 164

CHAPTER X

APPLICATIONS TO MECHANICS

78. Mass and density 167
79. Center of mass 168
80. Moment of inertia. Radius of gyration 173

CHAPTER XI

APPROXIMATE INTEGRATION. INTEGRATION BY MEANS OF SERIES. INTEGRATION BY MEANS OF THE MEASUREMENT OF AREAS

81. Approximate integration 177
82. Integration in series 177
83. Expansion of functions by means of integration in series . . 179
84. Evaluation of definite integrals by the measurement of areas . 181
85. The trapezoidal rule 182
86. The parabolic, or Simpson's one-third rule 184
87. Durand's rule 187
88. The planimeter 188

CHAPTER XII

INTEGRAL CURVES

89. Introduction 190
90. Special case of differentiation under the sign of integration . . 190
91. Integral curves defined. Their analytical relations . . . 192
92. Simple geometrical relations of integral curves 194

ART. PAGE
93. Simple mechanical relations and applications of integral curves. Successive moments of an area about a line 195
94. Practical determination of an integral curve from its fundamental curve. The integraph 198
95. The determination of scales 200

CHAPTER XIII

Ordinary Differential Equations

96. Differential equation. Order. Degree 201
97. Constants of integration. General and particular solutions. Derivation of a differential equation 202

Section I. Equations of the First Order and the First Degree

98. Equations in which the variables are easily separable . . . 205
99. Equations homogeneous in x and y 205
100. Exact differential equations 206
101. Equations made exact by means of integrating factors . . . 207
102. Linear equations 208
103. Equations reducible to the linear form 209

Section II. Equations of the First Order but not of the First Degree

104. Equations that can be resolved into component equations of the first degree 210
105. Equations solvable for y 211
106. Equations solvable for x 212
107. Clairaut's equation 213
108. Geometrical applications. Orthogonal trajectories . . . 214

Section III. Equations of an Order Higher than the First

109. Equations of the form $\frac{d^ny}{dx^n} = f(x)$ 218
110. Equations of the form $\frac{d^2y}{dx^2} = f(y)$ 218
111. Equations in which y appears in only two derivatives whose orders differ by unity 219
112. Equations of the second order with one variable absent . . 220
113. Linear equations. General properties. Complementary functions. Particular integral 222
114. The linear equation with constant coefficients and second member zero 223

ART. PAGE

115. Case of the auxiliary equation having equal roots 224

116. The homogeneous linear equation with the second member zero . 225

Examples on Chapter XIII. 227

APPENDIX

Note A. A method of decomposing a rational fraction into its partial fractions 229

Note B. To find reduction formulæ for $\int x^m(a + bx^n)^p\,dx$ by integration by parts 231

Note C. To find reduction formulæ for $\int \sin^m x \cos^n x\,dx$ by integration by parts 233

Note D. A theorem in the infinitesimal calculus 234

Note E. Further rules for the approximate determination of areas . 235

Note F. The fundamental theory of the planimeter 237

Note G. On integral curves 240

- 1. Applications to mechanics 240
- 2. Applications in engineering and in electricity . . 242
- 3. The theory of the integraph 244

Figures of some of the curves referred to in the examples . . . 246

A short table of integrals 249

Answers to the examples 263

Index 285

INTEGRAL CALCULUS

CHAPTER I

INTEGRATION A PROCESS OF SUMMATION

1. Uses of the integral calculus. Definition and sign of integration. The integral calculus can be used for two purposes, namely:

(*a*) To find the sum of an infinitely large number of infinitesimals of the form $f(x)\,dx$;

(*b*) To find the function whose differential or whose differential coefficient is given; that is, to find an anti-differential or an anti-derivative.

The integral calculus was invented in the course of an endeavor to calculate the plane area bounded by curves. The area was supposed to be divided into an infinitely great number of infinitesimal parts, each part being called *an element* of the area; and the sum of these parts was the area required. The process of finding this sum was called *integration*, a name which implies the combination of the small areas into a whole, and hence the sum itself was called the whole or *the integral*.

From the point of view of the first of the purposes just indicated, integration may be defined as *a process of summation*. In many of the applications of the integral calculus, and, in particular, in the larger number of those made by engineers, this is the definition to be taken. On the other hand, however, in many problems it is not a sum, but merely an anti-differential, that is required. For this purpose, integration may be defined as *an operation which is the inverse of differentiation*. It may at once be stated that in the course of making a summation by means of the integral calculus it will be necessary to find the anti-differential of some function; and it may also be said at this point, that the anti-differential can be shown to be the result of making a summation. Each of the above definitions of integration can be derived from the other. These statements will be found verified in Arts. 4, 11, 13.

In the differential calculus, the letter d is used as the symbol of differentiation, and $d\,f(x)$ is read "the differential of $f(x)$." In the integral calculus the symbol of integration is* $\int$, and $\int f(x)\,dx$ is read "the integral of $f(x)\,dx$." The signs d and $\int$ are signs of operations; but they also indicate the results of the operations of differentiation and integration respectively on the functions that are written after them.

The principal aims of this book are: (1) to explain how summations of infinitesimals of the form $f(x)\,dx$ may be made; (2) to show how the anti-differentials of some particular functions may be obtained.

2. Illustrations of the summation of infinitesimals. Two simple illustrations of the summation of an infinite number of infinitely small quantities will now be given. They will help to familiarize the student with a certain geometrical principle and with the fundamental theorem of the integral calculus, which are set forth in Arts. 3, 4. The method employed in these particular instances is identical with that used in the general case which follows them.

---

* This is merely the long S, which was used as a sign of summation by the earlier writers, and meant "the sum of." The sign $\int$ was first employed in 1675, and is due to Gottfried Wilhelm Leibniz (1646–1716), who invented the differential calculus independently of Newton. The word *integral* appeared first in a solution of James Bernoulli (1654–1705), which was published in the *Acta Eruditorum*, Leipzig, in 1690. Leibniz had called the integral calculus *calculus summatorius*, but in 1696 the term *calculus integralis* was agreed upon between Leibniz and John Bernoulli (1667–1748). See Cajori, *History of Mathematics*, pp. 221, 237.

(a) Find the area between the line whose equation is $y = mx$, the x-axis, and the ordinates for which $x = a$, $x = b$.

Let OL be the line $y = mx$; let OA be equal to a, and OB to b, and draw the ordinates AP, BQ. It is required to find the area of $APQB$. Divide the segment AB into n parts, each equal to Δx; and at the points of section A_1, A_2, $\cdots$, erect ordinates A_1P_1, A_2P_2, $\cdots$, which meet OL in P_1, P_2, $\cdots$. Through P, P_1, P_2, $\cdots$, Q, draw lines parallel to the axis of x and intersecting the nearest ordinate on each side, as shown in Fig. 1, and produce PB_1 to meet BQ in C.

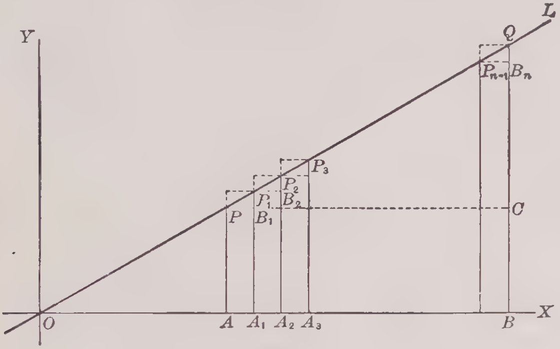

Fig. 1.

It will first be shown that the area $APQB$ is the limit of the sum of the areas of the rectangles PA_1, P_1A_2, $\cdots$, when n, the number of equal divisions of AB, approaches infinity, or, what is the same thing, when Δx approaches zero. The area $APQB$ is greater than the sum of the "inner" rectangles PA_1, P_1A_2, $\cdots$; and it is less than the sum of the "outer" rectangles AP_1, A_1P_2, $\cdots$. The difference between the sum of the inner rectangles and the sum of the outer rectangles is equal to the sum of the small rectangles PP_1, P_1P_2, $\cdots$.

The latter sum is equal to

$$B_1P_1\,\Delta x + B_2P_2\,\Delta x + \cdots + B_nQ\,\Delta x;$$

that is, to $(B_1P_1 + B_2P_2 + \cdots + B_nQ)\,\Delta x$, which is $CQ\,\Delta x$.

This may be briefly expressed,

$$\Sigma AP_1 - \Sigma PA_1 = \Sigma PP_1$$
$$= CQ\,\Delta x.$$

When Δx is an infinitesimal, the second member of this equation is also an infinitesimal of the first order; therefore, when Δx is infinitely small the limit of the difference between the total areas of the inner and of the outer rectangles is zero. The area $APQB$ lies between the total area of all of the inner and the total area of all of the outer rectangles. Hence, the area $APQB$ is the limit both of the sum of the inner rectangles and of the sum of the outer rectangles as Δx approaches zero. Each elementary rectangle has the area $y\,\Delta x$, that is $mx\,\Delta x$, since $y = mx$. The altitudes of the successive inner rectangles, going from A towards B, are ma, $m(a+\Delta x)$, $m(a+2\,\Delta x)$, $\cdots$, $m(a+(n-1)\,\Delta x)$. Hence,

$$\text{Area } APQB = \text{limit}_{\Delta x \doteq 0} m\{a\,\Delta x + (a+\Delta x)\,\Delta x + (a+2\,\Delta x)\,\Delta x + \cdots$$
$$+ (a + \overline{n-1}\,\Delta x)\,\Delta x\}\text{[*]}$$
$$= \text{limit}_{\Delta x \doteq 0} m\{a + (a+\Delta x) + (a+2\,\Delta x) + \cdots$$
$$+ (a + \overline{n-1}\,\Delta x)\}\,\Delta x.$$

Addition of the arithmetic series in brackets gives

$$\text{Area } APQB = \text{limit}_{\Delta x \doteq 0} \frac{mn\,\Delta x}{2}\{2\,a + (n-1)\,\Delta x\}$$
$$= \text{limit}_{\Delta x \doteq 0} \frac{m(b-a)}{2}\{b + a - \Delta x\}, \text{ since } n\,\Delta x = b - a,$$
$$= m\left(\frac{b^2}{2} - \frac{a^2}{2}\right).$$

---

[*] The symbol $\Delta x \doteq 0$ means "when Δx approaches zero as a limit." It is due to the late Professor Oliver of Cornell University.

In this example the element of area is obtained by taking a rectangle of altitude y, that is, mx, and width Δx, and then letting Δx become infinitesimal.

The expression $n \doteq \infty$ may be used instead of $\Delta x \doteq 0$, since

$$\Delta x = \frac{b-a}{n}.$$

It may be noted in passing, that if the anti-differential of $mx\,dx$, namely $\frac{mx^2}{2}$, be taken, and b and a be substituted in turn for x, the difference between the resulting values will be the expression obtained above.

(*b*) Find the area between the parabola $y = x^2$, the x-axis, and the ordinates for which $x = a$, $x = b$.

Let Q_1OQ be the parabola $y = x^2$; let OA be equal to a, and OB to b. Draw the ordinates AP, BQ. It is required to find the area $APQB$. Divide the segment AB into n parts each equal to Δx, and at the points of division A_1, A_2, $\cdots$, erect ordinates A_1P_1, A_2P_2, $\cdots$. Through P, P_1, P_2, $\cdots$, draw lines parallel to the axis of x and intersecting the nearest ordinates on each side, as in Fig. 2. It can be shown, in the same way as in the previous illustration, that the area $APQB$ is equal to the limit of the sum of

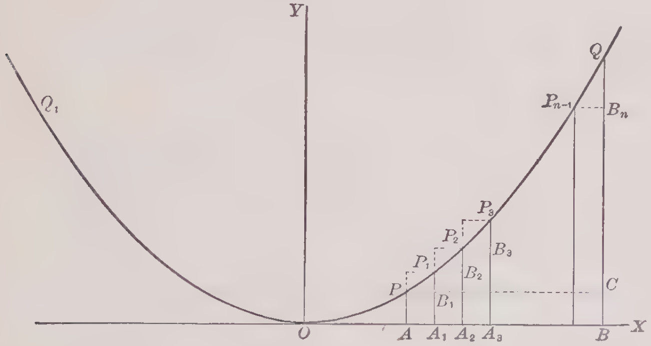

Fig. 2.

the rectangles PA_1, P_1A_2, $\cdots$, $P_{n-1}B$, when Δx approaches zero. This area will now be calculated.

The area of any elementary rectangle is $y\,\Delta x$, that is $x^2\,\Delta x$, since $y = x^2$; and the altitudes of the successive rectangles, going from A toward B, are a^2, $(a + \Delta x)^2$, $(a + 2\,\Delta x)^2$, $\cdots$. Hence,

$$\begin{aligned}
\text{Area } APQB &= \text{limit}_{\Delta x \doteq 0}\{a^2\,\Delta x + (a + \Delta x)^2\,\Delta x + (a + 2\,\Delta x)^2\,\Delta x + \cdots \\
&\qquad\qquad + (a + \overline{n-1}\,\Delta x)^2\,\Delta x\} \\
&= \text{limit}_{\Delta x \doteq 0}\{a^2 + (a + \Delta x)^2 + (a + 2\,\Delta x)^2 + \cdots \\
&\qquad\qquad + (a + \overline{n-1}\,\Delta x)^2\}\,\Delta x \\
&= \text{limit}_{\Delta x \doteq 0}\{na^2 + 2\,a\,\Delta x(1 + 2 + 3 + \cdots + \overline{n-1}) \\
&\qquad\qquad + (\Delta x)^2(1^2 + 2^2 + 3^2 + \cdots + \overline{n-1}^2)\}\,\Delta x.
\end{aligned}$$

It is shown in algebra that the sum of the squares of the first n natural numbers, 1^2, 2^2, 3^2, $\cdots$, n^2, is $\dfrac{n(n+1)(2\,n+1)}{6}$. The application of this result to the sum of squares in the second member of the last equation and the summation of the arithmetical series 1, 2, 3, $\cdots$ $(n-1)$, gives

$$\begin{aligned}
\text{Area } APQB = \text{limit}_{\Delta x \doteq 0}\,n\,\Delta x \Big\{ a^2 + an\,\Delta x - a\,\Delta x \\
+ (\Delta x)^2 \frac{(n-1)(2\,n-1)}{6} \Big\}.
\end{aligned}$$

But $$n\,\Delta x = b - a;$$

and hence, $$a^2 + an\,\Delta x + \frac{n^2}{3}(\Delta x)^2 = \tfrac{1}{3}(a^2 + ab + b^2).$$

Hence,
$$\begin{aligned}
APQB &= \text{limit}_{\Delta x \doteq 0}(b - a)\Big\{\frac{a^2 + ab + b^2}{3} - a\,\Delta x \\
&\qquad + \frac{1}{6}(\Delta x)^2 - \frac{b-a}{2}\Delta x\Big\} \\
&= \frac{b^3}{3} - \frac{a^3}{3}.
\end{aligned}$$

In this example, the element of area is obtained by taking a rectangle whose area is $x^2\Delta x$ and then letting Δx be an infinitesimal. Here also, it may be noted in passing, that if the anti-differential of x^2dx, namely $\frac{x^3}{3}$, be taken, and b and a substituted in turn for x, the difference between the resulting values will be the expression just derived for the area.

3. Geometrical principle. Let $f(x)$ be a continuous function of x, and let PQ be an arc of the curve whose equation is $y=f(x)$. Draw the ordinates AP, BQ, and suppose $OA=a$, $OB=b$. It is required to find the area $APQB$; that is, the area between the curve, the x-axis, and the ordinates AP, BQ.

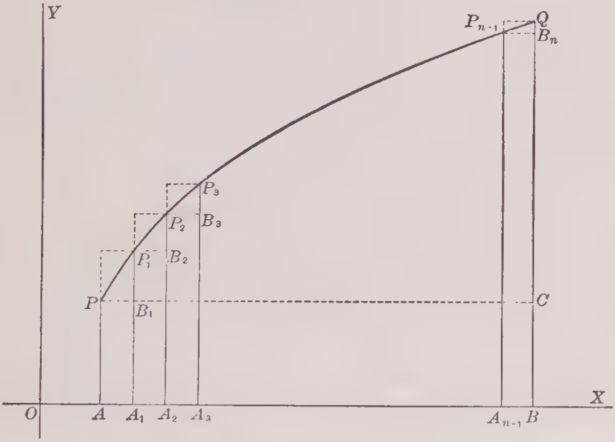

FIG. 3.

Divide the segment AB into n parts, each equal to Δx, and at the points of section A_1, A_2, $\cdots$, erect the ordinates A_1P_1, A_2P_2, $\cdots$ to meet the curve in P_1, P_2, $\cdots$. Through P, P_1, P_2, $\cdots$, draw lines parallel to the x-axis and intersecting the nearest ordinate on each side, as in Fig. 3, and produce PB_1 to meet BQ in C.

It will now be shown that the area $APQB$ is the limit of the sum of the areas of the rectangles PA_1, P_1A_2, $\cdots$, when the number of equal divisions of AB is made infinitely great, that is, when Δx is made infinitely small. The area $APQB$ is greater than the sum of the areas of the inner rectangles PA_1, P_1A_2, $\cdots$, and less than the sum of the areas of the outer rectangles AP_1, A_1P_2, $\cdots$. That is,

$$\Sigma PA_1 < APQB < \Sigma P_1A.$$

The difference between the sum of the outer rectangles and the sum of the inner rectangles is equal to the sum of the small rectangles PP_1, P_1P_2, $\cdots$; that is,

$$\begin{aligned}\Sigma P_1A - \Sigma PA_1 &= \Sigma PP_1\\ &= B_1P_1\Delta x + B_2P_2\Delta x + \cdots + B_nQ\,\Delta x\\ &= (B_1P_1 + B_2P_2 + \cdots + B_nQ)\Delta x\\ &= CQ\,\Delta x.\end{aligned}$$

The difference, $CQ\,\Delta x$, can be made as small as one pleases by decreasing Δx. Therefore, since the area of $APQB$ always lies between the areas of the inner and outer series of rectangles, and since the difference between these areas approaches zero as its limit, the area $APQB$ is the limit both of the sum of the inner rectangles, and of the sum of the outer rectangles.

Therefore, *the area included between the curve whose equation is* $y = f(x)$, *the x-axis, and a pair of ordinates, is the limit of the sum of the areas of the rectangles whose bases are successive segments of the part of the x-axis intercepted by the pair of ordinates, and whose altitudes are the ordinates erected at the points of division of the x-axis, as the bases approach zero.*

4. Fundamental theorem. Definite integral. Since the equation of the curve, an arc of which is given in Fig. 3, is $y = f(x)$, the heights of the successive inner rectangles, going towards the right from A, are

$$f(a),\ f(a + \Delta x),\ f(a + 2\,\Delta x),\ \cdots, f(a + (n-1)\,\Delta x).$$

Hence,

$$\text{Area } APQB = \text{limit}_{\Delta x \doteq 0} \{ f(a)\,\Delta x + f(a + \Delta x)\,\Delta x$$
$$+ f(a + 2\,\Delta x)\,\Delta x + \cdots + f(a + \overline{n-1}\,\Delta x)\,\Delta x \}. \tag{1}$$

The second member of (1) is the limit of the sum of the values, infinite in number, that $f(x)\,\Delta x$ takes as x varies by equal increments Δx from $x = a$ to $x = b$, when Δx is made infinitesimal. This limit may be indicated by

$$\textbf{Limit}_{\Delta x \doteq 0} \sum_{x=a}^{x=b} f(x)\,\Delta x.$$

In the integral calculus this is more briefly indicated by prefixing to $f(x)\,dx$ the sign $\int$, at the bottom and top of which are respectively written the values of x at which the summation begins and ends; thus:

$$\int_{x=a}^{x=b} f(x)\,dx.$$

An abbreviation for this form is

$$\int_a^b f(x)\,dx. \tag{2}$$

This is read, "the integral of $f(x)\,dx$ between the limits a and b." The initial and final values of x, namely, a and b, are called the *lower* and *upper* limits respectively of the integral.* The differential $f(x)\,dx$ is called *an element* of the integral. It evidently represents the area of any one of the component infinitesimal rectangles of altitude $f(x)$ and infinitesimal base dx. In the same way that dx is a differential of the distance along the x-axis, so $f(x)\,dx$, or $y\,dx$, as it is usually written, is a differential of the area between the curve and the axis of x.

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* This manner of indicating the limits between which the summation is to be made by writing the lower limit at the bottom and the upper limit at the top of the integration sign, is due to Joseph Fourier (1768–1830).

The limit of the sum in the second member of (1), which is indicated by the symbol (2), will now be obtained. Suppose that $f(x)\,dx$ is $d\phi(x)$, that is, suppose that $\phi(x)$ is the *anti-differential* of $f(x)\,dx$. Then, by the fundamental principle of the differential calculus,

$$\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}=f(x)+e,$$

in which e is a quantity that varies with x and approaches zero when Δx approaches zero. On clearing of fractions and transposing, the latter equation becomes

$$f(x)\,\Delta x=\phi(x+\Delta x)-\phi(x)-e\,\Delta x. \qquad (3)$$

On substituting in (3) the values of x at the successive points of division between A and B at intervals equal to Δx, the following equations are obtained:

$$\begin{aligned} f(a)\,\Delta x &= \phi(a+\Delta x)-\phi(a)-e_0\,\Delta x,\\ f(a+\Delta x)\,\Delta x &= \phi(a+2\,\Delta x)-\phi(a+\Delta x)-e_1\,\Delta x,\\ f(a+2\,\Delta x)\,\Delta x &= \phi(a+3\,\Delta x)-\phi(a+2\,\Delta x)-e_2\,\Delta x,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ f(b-\Delta x)\,\Delta x &= \phi(b)-\phi(b-\Delta x)-e_{n-1}\,\Delta x, \end{aligned}$$

in which each of the e's approaches zero when Δx approaches zero. The sum of the first members of these equations is equal to the sum of the second members; that is,

$$\sum_{x=a}^{x=b} f(x)\Delta x=\phi(b)-\phi(a)-(e_0+e_1+\cdots+e_{n-1})\,\Delta x.$$

Of the quantities e_0, e_1, $\cdots e_{n-1}$, suppose that e_i has an absolute value E not less than that of any one of the others. If E is substituted for each of the e's, the term

$$(e_0 + e_1 + \cdots + e_{n-1})\,\Delta x$$

becomes
$$nE\,\Delta x, \text{ or } (b-a)\,E,$$

since $n\,\Delta x = b - a$. Hence,

$$\sum_{x=a}^{x=b} f(x)\,\Delta x = \phi(b) - \phi(a) - \left\{\begin{array}{l}\text{a quantity which is not greater}\\ \text{than } (b-a)\,E\text{, and which ap-}\\ \text{proaches zero when } E \text{ ap-}\\ \text{proaches zero, that is, when}\\ \Delta x \text{ approaches zero.}\end{array}\right\}$$

Therefore, on letting Δx approach zero, there will be obtained,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Hence, *the sum or integral,* $\int_a^b f(x)\,dx$, *which is the sum of all the values, infinite in number, that* $f(x)\,dx$ *takes as* x *varies by infinitesimal increments from* a *to* b, *is found by obtaining the anti-differential* $\phi(x)$ *of* $f(x)\,dx$, *and subtracting the value of* $\phi(x)$ *for* $x = a$ *from its value for* $x = b$. The following notation is used to indicate these operations:

$$\int_a^b f(x)\,dx = \Big[\phi(x)\Big]_a^b = \phi(b) - \phi(a).^* \qquad (4)$$

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* It will be shown in Art. 9, that if $d\phi(x) = f(x)\,dx$, the anti-differential of $f(x)\,dx$ is $\phi(x) + c$, in which c is an arbitrary constant. Hence, equation (4) should be written

$$\int_a^b f(x)\,dx = \Big[\phi(x) + c\Big]_a^b.$$

Since the same c is used when a and b are substituted for x, this becomes

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a),$$

as above.

The sum $\int_a^b f(x)\,dx$ is called a *definite integral* because it has a definite value, the limit of the series in the second member of (1). Since the evaluation of this definite integral is equivalent to the measurement of the area between the curve $y = f(x)$, the x-axis, and the ordinates at $x = a$, $x = b$, the area may be regarded as representing the integral. It follows from the result (4) that a definite integral may be regarded as either:

(1) The limit of the sum of an infinitely large number of infinitesimal quantities of the form $f(x)\,dx$ taken between certain limits; or,

(2) The difference of the values of the anti-differential of $f(x)\,dx$ at each of these limits.

If $f(x)$ is any continuous function of x, $f(x)\,dx$ has an anti-differential.* However, the deduction of the anti-differential is often impossible, and in any case, is less simple and easy than the process of differentiation.†

Many of the practical applications of the integral calculus, such as finding areas, lengths of curves, volumes and surfaces of solids, centers of gravity, moments of inertia, mass, weight, etc., consist in making summations of infinitely small quantities. The integral calculus adds these infinitesimal quantities together and gives the result. It has been observed that in order to obtain the sum of infinitesimal areas, etc., the anti-differential of some function is required. Accordingly, a considerable part of any book on the integral calculus is devoted to the exposition of methods for obtaining the anti-differentials of functions which frequently appear in the process of solving practical problems.

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* The truth of this statement, for all the ordinary functions, will appear in the sequel. A proof applicable to all forms of continuous functions is given in Picard, *Traité d'Analyse*, t. I., No. 4.

† The phrase "to find the anti-differential" means to deduce a *finite* expression for the anti-differential in terms of the well-known mathematical functions. In cases in which the anti-differential cannot be thus obtained, an approximate value of the definite integral can be found by the methods discussed in Arts. 84–88. A short inspection of these articles may be made now

5. Supplement to Art. 3. In proving the principle of Art. 3, the arc PQ in Fig. 3 was used. If the arc of the given curve has the form and position in Fig. 4, the proof of the principle is as set forth in Art. 3. If the arc has the form and position in Fig. 5, and thus has maximum and minimum values of the ordinate, the principle still holds. This can be seen by drawing the maximum and minimum ordinates that come between AP and BQ, and remarking that the principle holds for the several successive parts APP_1A_1, $A_1P_1P_2A$, $\cdots$.

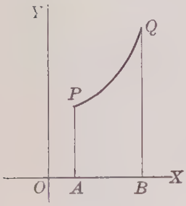

Fig. 4.

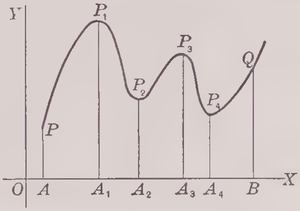

Fig. 5.

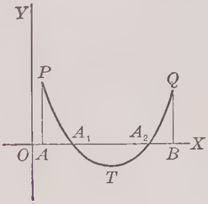

Fig. 6.

Suppose that the curve has the form and position in Fig. 6. The area of the part APA_1 is the limit of the sum of the elementary areas $f(x)\,\Delta x$ when Δx approaches zero and x varies from OA to OA_1; or, in other words, the area of APA_1 is the limit of the sum of the elementary areas $f(x)\,dx$ as x varies from OA to OA_1. Similarly, the area of A_1TA_2 is the limit of the sum of the elementary areas $f(x)\,dx$ as x varies from OA_1 to OA_2, and the area A_2QB is the limit of the sum of the elementary areas $f(x)\,dx$ as x varies from OA_2 to OB. In APA_1 and A_2QB, the ordinates that represent the values of $f(x)$ are positive, while in A_1TA_2 the ordinates are negative. Since x is taken as varying from left to right, dx is always positive. Accordingly, areas such as APA_1, A_2QB, which lie above the x-axis, have a positive sign, and areas such as A_1TA_2, which lie below the x-axis, have a negative sign. This example shows that in the case of a curve that crosses the x-axis, the method of summation by means of the integral calculus gives the *algebraic sum* of the areas that lie between the curve and the x-axis, the areas above the x-axis being given a positive sign, and those below receiving a negative sign.

If the total absolute area between the curve and the axis of x is required, the portions APA_1, A_1TA_2, A_2QB, should be found separately.

NOTE. If n is a constant not equal to -1, the anti-differential of $u^n\,du$ is $\frac{u^{n+1}}{n+1}$; for, differentiation of the latter gives $u^n\,du$.

Ex. 1. Find the area between the curve whose equation is $y = x^3$, the x-axis and the ordinates for which $x = 1$, $x = 4$.

By Art. 4, the area required $= \int_1^4 x^3\,dx$

$$\begin{aligned} &= \left[\frac{x^4}{4} + c\right]_1^4 \\ &= \tfrac{256}{4} + c - (\tfrac{1}{4} + c) \\ &= 63\tfrac{3}{4} \text{ units of area.} \end{aligned}$$

Ex. 2. Find the area between the parabola $2y = 5x^2$, the x-axis and the ordinates for which $x = 2$, $x = 5$. *Ans.* $97\frac{1}{2}$ square units.

Ex. 3. Find the area between the line $y = 4x$, the x-axis and the ordinates for which $x = 2$, $x = 11$. *Ans.* 234 square units.

Ex. 4. Find the area between the parabola $2y = 3x^2$, the x-axis and the ordinates for which $x = -3$, $x = 5$. *Ans.* 76 square units.

Ex. 5. Find the area between the line $y = 5x$, the x-axis and the ordinate for which $x = 2$. *Ans.* 10 square units.

Ex. 6. Find the area between the line $y = 5x$, the x-axis and the ordinates for which $x = -2$, $x = 2$. *Ans.* 0.

6. Geometrical representation of an integral. It is necessary to perceive clearly that a definite integral, whether it be the sum of an infinite number of infinitesimal elements of area, length, volume, surface, mass, force, work, etc., can be represented graphically by an area. For instance, in order to represent the definite integral

$$\int_a^b f(x)\,dx,$$

choose a pair of rectangular axes, plot the curve whose equation is

$$y = f(x),$$

and draw the ordinates for $x = a$, $x = b$. It has been shown in Art. 4 that the area between this curve, the x-axis, and these ordinates has the value of the definite integral above. Hence, this area can represent the integral. This does not mean that the area is equal to the integral, for the integral may be a length, a volume, etc. The area can be taken to represent the integral, because the *number* that indicates the area is equal to the *number* that indicates the value of the integral. That an integral may be represented geometrically by an area is at the foundation of some important theorems and applications of the integral calculus.

7. Properties of definite integrals. In Art. 4 it was shown that if

$$\frac{d\phi(x)}{dx} = f(x),$$

the definite integral, $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

From this, the first of the following properties is immediately deducible. The second and third properties depend upon Art. 6.

(*a*) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

This relation holds since the second member is $-\{\phi(a) - \phi(b)\}$; that is, $\phi(b) - \phi(a)$. Hence, the algebraic sign of a definite integral is changed by an interchange of the limits of integration.

(*b*) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Let the curve whose equation is $y = f(x)$ be drawn; and let ordinates AP, BQ, CR be erected at the points for which $x = a$, $x = b$, $x = c$. Since

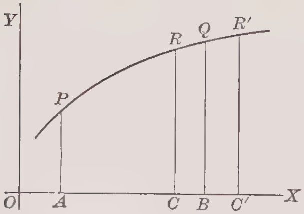

Fig. 7.

$$\text{area } APQB = \text{area } APRC + \text{area } CRQB,$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

It does not matter whether c is between a and b or not. For, suppose $OC' = c'$, and draw the ordinate $C'R'$; then

$$\text{area } APQB = \text{area } APR'C' - \text{area } BQR'C';$$

and hence, $$\int_a^b f(x)\,dx = \int_a^{c'} f(x)\,dx - \int_b^{c'} f(x)\,dx,$$

which, by (a), $$= \int_a^{c'} f(x)\,dx + \int_{c'}^b f(x)\,dx.$$

Therefore a given definite integral may be broken up into any number of similar definite integrals that differ only in the limits between which integration is to be performed.

(c) Construct $APQB$ as in (b) to represent the definite integral $\int_a^b f(x)\,dx$. Then

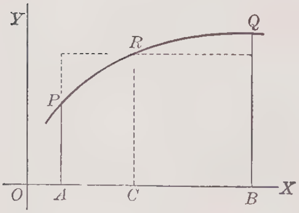

Fig. 8.

$$\int_a^b f(x)\,dx = \text{area } APQB$$

= area of a rectangle whose height CR is greater than AP and less than BQ, and whose base is AB,

$$= AB \cdot CR$$

$$= (b - a) f(c),$$

OC being equal to c.

Therefore $$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}.$$

The function $f(c)$ is called *the mean value* of $f(x)$ for values of x that vary continuously from a to b. This mean value may be defined to be equal to the height of a rectangle which has a base equal to $b-a$ and an area that is equivalent to the value of the integral. The subject of mean values is discussed further in Arts. 76, 77.

CHAPTER II

INTEGRATION THE INVERSE OF DIFFERENTIATION

8. Integration the inverse of differentiation. In Art. 1, two definitions of integration were indicated, namely:

(*a*) Integration is a process of summation;

(*b*) Integration is an operation which is the inverse of differentiation.

The first definition was discussed in Chapter I. In this and the next following article, integration will be considered from the point of view of the second definition.

The differential calculus is in part concerned with finding the differential or the derivative of a given function. On the other hand, the integral calculus is in part concerned with finding the function when its differential or its derivative is given. If a function be given, the differential calculus affords a means of deducing the rate of increase of the function per unit increase of the independent variable. If this rate of increase of a function be known, the integral calculus affords a means of finding the function.

Ex. 1. A curve whose equation is $y = 4x^2$ is given; and the rate of increase of the ordinate per unit increase of the abscissa is required.

Since $$y = 4x^2,$$

$$\frac{dy}{dx} = 8x.$$

This means that the ordinate at a point whose abscissa is x is increasing $8x$ times as fast as the abscissa. If this rate of increase remains uniform, the ordinate will receive an increase of $8x$ when the abscissa is increased by unity. This determines the direction of the curve at the point.

On the other hand, suppose that at any point on a curve, it is known that

$$\frac{dy}{dx} = 8x,$$

and let the equation of the curve be required. It evidently follows from this equation that

$$y = 4x^2 + c,$$

in which c is an arbitrary constant. The constant c can receive any one of an infinite number of values; and hence, the number of curves that satisfy the given condition is infinite. If an additional condition be imposed, for example, that the curve pass through the point (2, 3), then c will have a definite corresponding value. For, since the point (2, 3) is on the required curve,

$$3 = 4.2^2 + c,$$

and accordingly, $$c = -13.$$

Hence, the equation of the curve that satisfies both of the conditions above given is

$$y = 4x^2 - 13.$$

Ex. **2**. In the case of a body falling from rest under the action of gravity, the distance s through which it falls in t seconds is a constant, approximately 16 times t^2 feet; find the velocity at any instant.

Here, $$s = 16t^2,$$

and hence, $$\frac{ds}{dt} = 32t;$$

that is, the velocity * in feet per second at the end of t seconds is $32t$.

On the other hand, suppose it is known that in the case of a body falling from rest, the velocity in feet per second is 32 multiplied by the time in seconds since motion began. Let the corresponding relation between the distance and the time be required.

Here, it is known that $$\frac{ds}{dt} = 32t.$$

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* If a body moves in a straight line through a distance Δs in a time Δt, and if its average velocity be denoted by v,

$$v = \frac{\Delta s}{\Delta t}.$$

As Δt approaches zero, Δs also approaches zero, and the velocity approaches the definite limiting value $\frac{ds}{dt}$.

It is obvious that the solution of this simple differential equation is

$$(1)\quad s = 16t^2 + c,$$

in which c is an arbitrary constant. This result is indefinite. By the conditions of the question, however, $s = 0$ when $t = 0$.

Hence, substituting in (1), $0 = 0 + c$;

whence, $c = 0$,

and $s = 16t^2$

is the definite solution.

The distance through which the body falls can also be determined by the method of summation employed in the first chapter. Let the number 32 be denoted by g. The distance passed over in any time is equal to the product of the average velocity during that time and the time. The time t may be divided into n equal intervals Δt, so that $t = n\Delta t$. The velocity at the beginning of the rth interval is $(r-1)g\,\Delta t$, and at the end of the interval is $rg\,\Delta t$. Hence, the distance passed over in the interval lies between

$$(r-1)g(\Delta t)^2 \text{ and } rg(\Delta t)^2.$$

On finding similar limits for the distance passed over in the case of each of the intervals and adding, it will be found that the total distance passed over lies between

$$[0 + 1 + 2 + \cdots + (n-1)]g(\Delta t)^2 \text{ and } [1 + 2 + \cdots + n]g(\Delta t)^2;$$

that is, summing these arithmetical series, the distance passed over lies between

$$\frac{n(n-1)}{2}g(\Delta t)^2 \text{ and } \frac{n(n+1)}{2}g(\Delta t)^2.$$

Since $\Delta t = \frac{t}{n}$, the distance lies between

$$\frac{gt^2}{2} - \frac{gt^2}{2n} \text{ and } \frac{gt^2}{2} + \frac{gt^2}{2n};$$

and the distance is the common limit of these two expressions, when Δt approaches zero, that is, when n approaches infinity. Hence, $s = \frac{1}{2}gt^2$.

Sometimes the anti-differential (or the anti-derivative) of a function is wanted for its own sake alone, as in the illustrations given above; and sometimes it is desired for further ends, as, for example, in the process of making a summation by means of the integral calculus in Art. 4. The anti-differential is called the *integral*, the process of finding it is called *integration*, and the symbol of integration is the sign $\int$. Thus, if the differential of $\phi(x)$ is $f(x)\,dx$, which is expressed by

$$d\phi(x) = f(x)\,dx, \tag{1}$$

then

$$\int f(x)\,dx = \phi(x). \tag{2}$$

Equation (1) may be read "the differential of $\phi(x)$ is $f(x)\,dx$;" equation (2) may be read "the function of which the differential is $f(x)\,dx$, is $\phi(x)$." For brevity, the latter may be read "the integral of $f(x)\,dx$ is $\phi(x)$." *

Memory of the fundamental formulæ of differentiation will carry one far in the integral calculus. For instance, since dx^4 is $4\,x^3\,dx$, $\int 4\,x^3\,dx$ is x^4; since $d\sin x$ is $\cos x\,dx$, $\int \cos x\,dx$ is $\sin x$. The beginner will see the necessity of having ready command of the formulæ for differentiation, since they will be employed in the inverse process of integration.† Differentiation

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[*] The origin of the terms *integral*, *integration*, and of the sign $\int$ has been given in Art. 1. Instead of the sign $\int$, the symbols d^{-1} and D^{-1} are sometimes employed: thus, $d^{-1}f(x)\,dx$, which is read "the anti-differential of $f(x)\,dx$," and $D^{-1}f(x)$, which is read, "the anti-derivative of $f(x)$." In the case of the second definition of integration, the use of the symbols d^{-1}, D^{-1}, is more logical than the use of $\int$. The latter sign is, however, firmly established in this connection. It may be remarked that the differential is more frequently written than is the derivative of a function.

[†] The expressions $\int x^2\,dx$, $d^{-1}(x^2\,dx)$, $D^{-1}(x^2)$ are equivalent. The inverse process of integration is not always practically possible (see Art. 4). Art. 81 may be referred to for examples of differentials whose integrals cannot be expressed in a finite form.

of both members of (2) g.

$$d\int f(x)\,dx = d\phi(x),$$

whence, by (1), $$= f(x)\,dx.$$

Therefore, d neutralizes the effect of $\int$. It will be shown in the next article that $\int d\phi(x)$ may have values different from $\phi(x)$.

9. Indefinite integral. Constant of integration. Since $d(x^4 + c)$ is $4x^3\,dx$, c being any constant, $\int 4x^3\,dx$ is $x^4 + c$. The integral given in Art. 8 comes from this on making c zero. But c may be given any other value that does not involve x. Hence, $\int 4x^3\,dx$ is indefinite so far as an arbitrary added constant is concerned. In general,

if $$d\phi(x) = f(x)\,dx, \tag{1}$$

then $$\int f(x)\,dx = \phi(x) + c, \tag{2}$$

in which c is any constant; for differentiation of the members of (2) shows that $f(x)\,dx = d\phi(x)$. Hence, *the integral of a given differential is indefinite so far as an arbitrary added constant is concerned.* Illustrations have been seen in the preceding articles. It should be noted that the indefiniteness does not extend to terms that contain x. In other words, a given differential can have an infinite number of integrals that correspond to the infinite number of values that an arbitrary constant can take, but any two of these integrals differ only by a constant. For instance,

$$\int (x+1)\,dx = \frac{x^2}{2} + x + c_1.$$

But on substituting z for $x + 1$, and consequently, dz for dx,

$$\int (x+1)\,dx = \int z\,dz = \frac{z^2}{2} + c_2 = \frac{(x+1)^2}{2} + c_2 = \frac{x^2}{2} + x + \frac{1}{2} + c_2.$$

These two integrals agree in the terms that contain x. When an integration is performed, the arbitrary constant should be indicated in the result; or, if not indicated, it should be understood to be there. The second member of (2) is usually called the *indefinite integral* of $f(x)\,dx$, and c is said to be the constant of integration. When the constant of integration has an arbitrary value, that is, when no definite value has been assigned to it, the integral is called also a *general integral;* on the other hand, when the constant of integration is given a particular value, the integral is said to be a *particular integral.* For instance, the general integral (and indefinite integral) of $x^3\,dx$ is $\frac{1}{4}x^4 + c$ in which c is arbitrary. A particular integral of $x^3\,dx$ is obtained by giving c any one of an infinity of possible values, say $6, -5, \frac{1}{2}$. Thus $\frac{1}{4}x^4 + 6$, $\frac{1}{4}x^4 - 5$, $\frac{1}{4}x^4 + \frac{1}{2}$ are particular integrals. In practice the value of the constant may be determined by the special conditions of the problem.

10. Geometrical meaning of the arbitrary constant of integration.

If
$$\frac{dy}{dx} = F'(x), \tag{1}$$

then
$$y = \int F'(x)\,dx,$$

that is,
$$y = F(x) + c, \tag{2}$$

in which c is an arbitrary constant of integration. Suppose that c is given particular values, say $8, -3$, etc.; and let the curves whose equations are

$$\left.\begin{aligned} y &= F(x) + 8,\\ y &= F(x) - 3, \end{aligned}\right\} \tag{3}$$
$$\text{etc., etc.,}$$

be drawn. All of these curves have the same value of $\frac{dy}{dx}$; that is, the same direction, for the same value of x. Also, for any two of the curves the difference in the lengths of their ordinates remains the same for all values of x. For example, for any value of x, the difference between the lengths of the ordinates of the two curves whose equations are given in (3) is $8-(-3)$, or 11. Hence, all the curves, whose equations are of the form (2), thus differing merely in the c's, can be obtained by moving any one of the curves vertically up or down. The particular value assigned to c merely determines the position of the curve with respect to the x-axis, and has nothing to do with its form. Fig. 9 illustrates this.

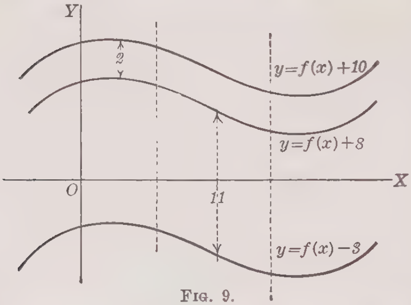

FIG. 9.

11. Relation between the indefinite and the definite integral. In Art. 4 it has been seen that if $d\phi(x)=f(x)\,dx$, the sum or integral of $f(x)\,dx$ for all values of x from $x=a$ to $x=b$, satisfies the relation

$$\int_a^b f(x)\,dx=\phi(b)-\phi(a). \tag{1}$$

If the upper limit be variable and be denoted by x,

$$\int_a^x f(x)\,dx=\phi(x)-\phi(a). \tag{2}$$

If the lower limit a be arbitrary, $-\phi(a)$ may be represented by an arbitrary constant c, and (2) becomes

$$\int_a^x f(x)\,dx=\phi(x)+c. \tag{3}$$

But

$$\int f(x)\,dx=\phi(x)+c. \tag{4}$$

Hence, an indefinite integral is an integral whose upper limit is the variable and whose lower limit is arbitrary. The first member of (4) may be considered an abbreviation for the first member of (3). The indefinite integral may therefore be regarded as obtained by a process of summation.

12. Examples that involve anti-differentials. This article is inserted for the purpose of giving simple, typical examples of a practical kind, in which anti-differentials are required for purposes other than that of summation. These illustrations will also involve the determination of constants. In many applications of the calculus, *two kinds of constants* must be distinguished, namely, those which are constants of integration, and those which are given distances, angles, forces, etc.; for example, the constant g, in Ex. 2, Art. 8, and h, b, k, a, in Ex. 2 below. Rectangular coördinates are used in the following exercises.

Ex. 1. Determine the equation of the curve at every point of which the tangent has the slope $\frac{1}{2}$. Determine the equation of the curve which passes through the point (2, 3) and also satisfies the former condition. The slope of a curve $y = f(x)$ at any point (x, y) is $\frac{dy}{dx}$.* Hence, by the given condition,

$$(1) \quad \frac{dy}{dx} = \frac{1}{2}.$$

Adopting the differential form, $dy = \frac{1}{2}\,dx$,

and integrating, $$(2) \quad y = \tfrac{1}{2}x + c,$$

the equation of a straight line. Now c, the arbitrary constant of integration, which in this case represents the intercept of the line on the y-axis, can take an infinite number of values. The first condition is therefore satisfied by each and all of the parallel lines of slope $\frac{1}{2}$.

If, in addition, the line is required to pass through the point (2, 3), then $x = 2$, $y = 3$, satisfy (2), and $3 = \frac{1}{2} \cdot 2 + c$. From this, $c = 2$. Hence, the curve that satisfies both of the given conditions is the line whose equation is

$$y = \tfrac{1}{2}x + 2.$$

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* By the slope of a curve at any point is meant the tangent of the angle that the tangent line to the curve at the point makes with the x-axis.

Ex. 2. Determine the equation of the curve that shall have a constant subnormal. Also, determine the curve which has a constant subnormal and passes through the two given points (o, h), (b, k); and find the length of its constant subnormal.

Let A, B, be the given points (o, h), (b, k), and let P be any point (x, y) on the curve. Suppose that PT is the tangent at P, and PN the normal. Put the constant subnormal MN equal to a.

Since the angle $\alpha =$ angle θ (see Fig. 10), their sides being respectively perpendicular,

$$\tan \alpha = \tan \theta;$$

that is, (1) $$\frac{dy}{dx} = \frac{a}{y}.$$

Putting this in the differential form,

(2) $$y\,dy = a\,dx,$$

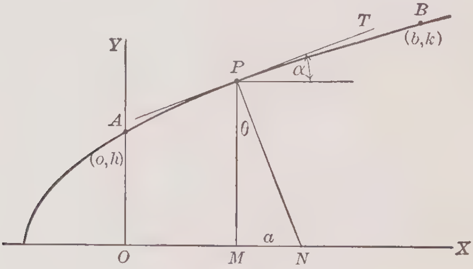

FIG. 10.

and integrating both sides, $$\frac{y^2}{2} + c' = ax + c'',$$

whence, (3) $$\frac{y^2}{2} = ax + c,$$

in which c', c'', are the constants of integration, and c denotes $c'' - c'$. Equation (3) is the equation of a parabola, and it includes an infinite number of parabolas, one for each of the infinite number of values that the arbitrary constant c can have.

The particular curve which passes through the points (o, h) (b, k), and has a constant subnormal is also required. Since the coördinates of these points must satisfy (3), it follows that

$$\frac{h^2}{2} = c,$$

and $$\frac{k^2}{2} = ab + c.$$

These equations suffice to determine c and the length a of the constant subnormal. On solving them, it is found that

$$c = \tfrac{1}{2} h^2, \quad a = \frac{k^2 - h^2}{2b}.$$

Hence, the equation of the second curve required is

$$by^2 = (k^2 - h^2)\, x + bh^2.$$

In the differential calculus it is shown that the length of the subnormal is $y \dfrac{dy}{dx}$. The first condition might have been expressed immediately by the equation

$$y \frac{dy}{dx} = a,$$

which is equivalent to (1) and (2).

Ex. **3**. Find the curve whose subtangent is constant and equal to a. Determine the curve so that it shall pass through the point (0, 1).

Ex. **4**. Find the curve for which the length of the subnormal is proportional to (say k times) the length of the abscissa.

13. Another derivation of the integration formula for an area. In Arts. 3, 4, it was shown that the area included between the curve $y = f(x)$, the x-axis, and the ordinates for $x = a$, $x = b$ is the limit of the sum of the infinitely large number of infinitesimal quantities $f(x)\,dx$, which are successively obtained as x varies continuously from a to b, and this limit was represented by the definite integral $\int_a^b f(x)\,dx$. The area can also be derived by means of the second definition of integration.

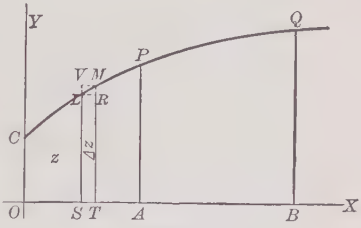

FIG. 11.

Let CPQ be an arc of the curve whose equation is $y = f(x)$, and let $OA = a$, $OB = b$. Draw the ordinates AP, BQ. Take any point S on the x-axis at a distance x from O, and draw the ordinate SL whose length is $f(x)$. Let z denote the area of $OCLS$. If x or OS is increased by ST, which is equal to Δx, and the ordinate TM be drawn, the area z will be increased by the area $SLMT$. This increment will be denoted by Δz. Draw LR parallel to the x-axis, and complete the rectangle $LRMV$ in which $LR = \Delta x$ and $RM = \Delta y$.

The increment of the area,

$$\begin{aligned}\Delta z &= SLMT \\ &= \text{rectangle } SLRT + \text{area } LRM \\ &= SLRT + \text{an area less than the rectangle } LRMV \\ &= f(x)\,\Delta x + \text{area less than } \Delta y \cdot \Delta x.\end{aligned}$$

Hence,

$$\frac{\Delta z}{\Delta x} = f(x) + \text{something less than } \Delta y.$$

When Δx approaches zero, Δy also approaches zero; and hence, in the limit,

$$\frac{dz}{dx} = f(x); \tag{1}$$

that is,

$$\frac{dz}{dx} = y. \tag{2}$$

Equation (2) means that the numerical value of the differential coefficient with respect to the abscissa, of the area between a curve, the axes of coördinates, and an ordinate, is the same as the numerical value of this ordinate of the curve.

Equations (1) and (2) written in the differential form give the differential of this area, namely,

$$dz = f(x)\,dx, \text{ and } dz = y\,dx. \tag{3}$$

Finding the anti-differentials in (3) gives as the area $OCLS$,

$$\begin{aligned}z &= \int f(x)\,dx \\ &= \phi(x) + c, \end{aligned} \tag{4}$$

in which $\phi(x)$ is the anti-derivative of $f(x)$ and c is an arbitrary constant. If the area is measured from the y-axis, the area is zero when x is zero. Hence, substituting these values in (4)

$$0 = \phi(0) + c,$$

whence, $$c = -\phi(0),$$

and (4) becomes $$z = \phi(x) - \phi(0).$$

Hence, $$\text{area of } OCPA = \phi(a) - \phi(0).$$

Similarly, $$\text{area of } OCQB = \phi(b) - \phi(0).$$

Since $$APQB = OCQB - OCPA,$$

it follows that $$\text{area } APQB = \phi(b) - \phi(a).$$

The expression in the second member is the same that was found for the area in Art. 4 by means of the first definition of integration.

If the area is measured, not from the y-axis, but from another fixed vertical line, say the ordinate at $x = m$, the derivation of equations (1) and (2) is the same as given above. In this case, the area is zero when $x = m$, and hence,

$$0 = \phi(m) + c. \quad \text{From this, } c = -\phi(m).$$

The value of c in (4) thus depends solely upon the fixed ordinate from which the area is measured.

14. A new meaning of y in the curve whose equation is $y = f(x)$. Derived curves. It has been seen in the differential calculus that in the case of a curve whose equation is $y = f(x)$, at any point on the curve the slope of the curve is $\frac{dy}{dx}$, the differential coefficient of its ordinate with respect to its abscissa. Art. 13, with equations (2) and (4), shows that at any point of a curve the length of the ordinate y is the differential coefficient with respect to the abscissa, of the area bounded by the curve, the axis of x, a fixed ordinate, and the ordinate at the point.* Therefore, "if we wish to make a graphic picture of any function and its derivative, we can represent the function either by the ordinate y of a curve or by its area, while its derivative will then be represented by its slope or ordinate respectively. If we are most interested in the *function*, we usually employ the former method (in which the ordinate represents the function); if in its *derivative*, the latter (in which the ordinate represents the derivative). That is, we usually like to use the *ordinate* to represent the main variable under consideration." †

For instance, suppose it is necessary to represent the function $f(x)$. Let the curve be drawn whose equation is

$$y = f(x). \tag{1}$$

At any point (x, y) on the curve, the ordinate y represents the value of the function for the corresponding value of x; and the slope $\frac{dy}{dx}$ represents the rate of change of the function compared with the rate of change of the variable x. Now let the curve be drawn whose equation is

$$y = f'(x), \tag{2}$$

in which $f'(x)$ denotes $\frac{df(x)}{dx}$. At any point (x, y) on this curve, the numerical measure of the ordinate is the same as that of the slope of the first curve at the point having the same abscissa. Hence, an ordinate at any point of the second curve represents the rate of change of the function $f(x)$ compared with the rate of change of the variable x at this point. Also, the area between the second curve, the axes, and the ordinate at (x, y) is

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* The remaining part of this article is not necessary for the articles that follow. However, it may be useful for the beginner to read it, because it may help to strengthen his grasp on the fundamental principles of the integral calculus.

† Irving Fisher, *A brief introduction to the Infinitesimal Calculus designed especially to aid in reading mathematical economics and statistics*, Art. 89. Some readers may be interested in an application of the principle quoted above. Professor Fisher continues: "Jevons, in his *Theory of Political Economy*, used the abscissa x to represent commodity, and the area z to represent its total utility, so that its ordinate y represented 'marginal utility' (*i.e.* the differential quotient of total utility with reference to commodity). Anspitz and Lieben, on the other hand, in their *Untersuchungen über die Theorie des Preises*, represent total utility by the ordinate and marginal utility by the *slope* of their curve."

$$\int_0^x f'(x)\, dx,$$

that is
$$f(x) - f(0).$$

Hence, the area of the curve $y = f'(x)$ bounded as described above *plus* a constant quantity $f(0)$ can represent the function $f(x)$.

For example, suppose that the function is $px^2 + 4$. That is,

$$f(x) = px^2 + 4,$$

and
$$f'(x) = 2\,px.$$

The parabola $y = px^2 + 4$, and the line $y = 2\,px$ are shown in Fig. 12. At any point M on the x-axis, the ordinates MP, MQ

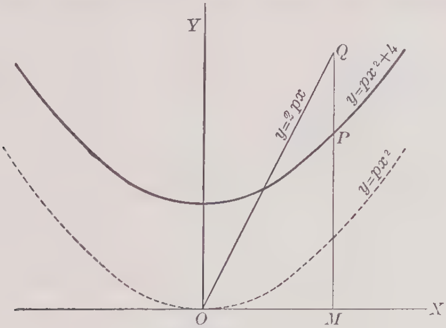

FIG. 12.

are drawn to these curves. The ordinate MP represents the function for $x = OM$; and the slope at P represents the rate of change of the function when $x = OM$. The ordinate MQ is equal (numerically) to the slope at P; and hence, it also represents the rate of change of the function when $x = OM$. Moreover, since

$$\text{area } OQM = \int_0^x 2\,px\,dx = px^2,$$

the function $f(x)$ for $x = OM$ is represented by the area $OQM + 4$. Had the function been px^2 (shown by the dotted curve), the area OQM would exactly represent the function.

To recapitulate: In the case of a function $f(x)$, if the curve

$$y = f(x), \tag{1}$$

and its first derived curve $y = f'(x)$, (2)

be drawn, the rate of change of the function for any value of x is represented equally well by the slope of the first curve and by the ordinate of the derived curve for that value of x; and the function itself for any value of x is represented equally well by the corresponding ordinate of the primary curve and by the area of the derived curve increased by the constant quantity $f(0)$.

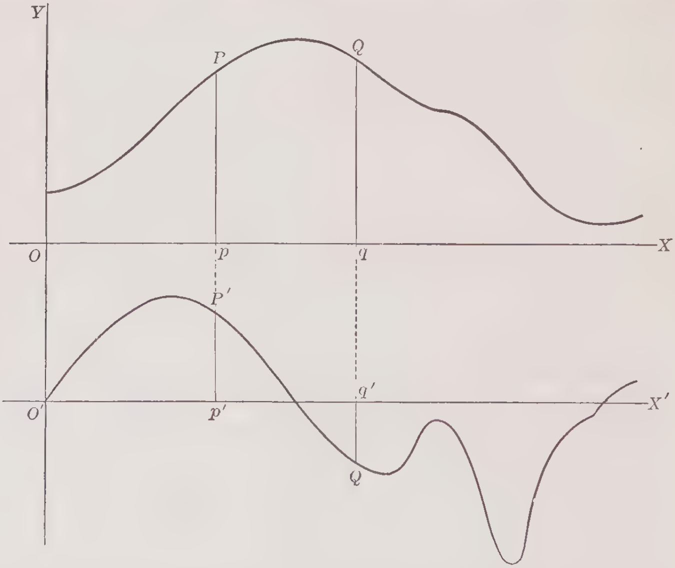

Fig. 18.

The derived curve (2) is called the "curve of slopes" of the first curve. Two of these curves are shown in Fig. 13. The horizontal scale is the same for both curves; but the ordinates on the original curve represent lengths while the ordinates on the derived represent tangents of angles. At a point at which the original curve has a maximum or a minimum ordinate, the slope is zero; and hence, the corresponding ordinate on the derived curve is zero. Conversely, when the derived curve crosses the x-axis, the corresponding ordinate of the original curve is a maximum or a minimum.

15. Integral curves. Let the curve whose equation is

$$y = f(x) \tag{1}$$

be drawn. Suppose that the anti-derivative of $f(x)$ is $\phi(x)$; and draw the curve whose equation is

$$y = \int_0^x f(x)\,dx, \tag{2}$$

that is, $$y = \phi(x) - \phi(0),$$

or, briefly, $$y = F(x). \tag{3}$$

The curve whose equation is (2) or (3) is called the *first integral curve* of the curve (1). It is evident that

$$\frac{dF(x)}{dx} = \frac{d\phi x}{dx} = f(x). \tag{4}$$

The following important properties can be deduced from equations (1), (2), (4).

(a) For the same abscissa x, the number that indicates the *length* of the ordinate of the first integral curve is the same as the number that indicates the *area* between the original curve, the axes, and ordinate for this abscissa. Therefore, the ordinates of the first integral curve can represent the areas of the original curve bounded as above described.

(*b*) For the same abscissa x, the number that indicates the slope of the first integral curve is the same as the number that indicates the length of the ordinate of the original curve. Therefore the ordinates of the original curve can represent the slopes of the first integral curve.

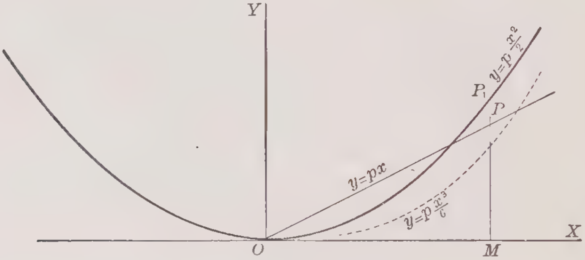

Fig. 14.

Example. The line whose equation is

$$y = px$$

has for its first integral curve the parabola whose equation is

$$y = \int_0^x px\,dx,$$

that is,
$$y = p\,\frac{x^2}{2}.$$

At any point M on the x-axis, OM being equal to x_1, say, erect the ordinates MP, MP_1, to the line and the parabola. The same number, namely $p\frac{x_1^2}{2}$, indicates both the length of the ordinate MP_1 and the area OPM; and the same number, namely px_1, indicates both the length of the ordinate MP and the slope of the tangent at P_1. This is true for the pair of ordinates erected at every point on the x-axis.

In like manner the curve whose equation is (2) has a first integral curve. The latter is called the second integral curve for the curve of equation (1). This second integral curve has a first integral curve which is called the third integral curve of (1), and so on. There is thus a series of successive integral curves for any given curve. For instance, the second integral curve of the line $y = px$ is the curve whose equation is

$$y = \int_0^x p\frac{x^2}{2},$$

that is, $$y = p\frac{x^3}{6}.$$

This curve is shown in the figure. The subject of successive integral curves has very important applications in problems in mechanics and engineering. Accordingly, an exposition of their properties and uses is given in Chapter XII.

16. Summary. This and the preceding chapter have been concerned with showing by statement and examples that integration may be regarded in two ways:

(1) As a process of summation, in which $\int_a^b f(x)\,dx$ denotes the limit of the sum indicated by $\sum\limits_{x=a}^{x=b} f(x)\Delta x$, when Δx approaches zero;

(2) As an operation which is the inverse of differentiation, in which $\int f(x)\,dx$ denotes $d^{-1}[f(x)\,dx]$ or $D^{-1}f(x)$; that is, denotes the anti-differential of $f(x)\,dx$, or, what is the same thing, the anti-derivative of $f(x)$.

It may be remarked that *the rules of integration* are all derived from the latter point of view. Both of these conceptions of integration are employed in problems in geometry, mechanics, and other subjects. The first view of integration is necessary to a clear understanding of the application of the integral calculus to the solution of certain problems; and, on the other hand, the second view is necessary to a clear understanding of the use of the calculus in the solution of certain other problems.

CHAPTER III

FUNDAMENTAL RULES AND METHODS OF INTEGRATION

17. In Chapters I. and II., the two purposes of integration were set forth; and definitions of integration based upon these purposes were given with illustrative examples. Relations between the definitions were also pointed out, particularly in Arts. 11, 13. It was also shown that in the process of making an integration, whatever the object may be, it is necessary to find an anti-differential or an anti-derivative of some function. A general method of differentiation is given in the differential calculus. Unfortunately, no general method for the inverse process of integration exists. It is necessary to derive a *rule* for the integration of each function. The formulæ of integration are derived or disclosed by falling back upon our knowledge of the rules of differentiation. In fact, the first simple rules, given in Art. 18, are merely directions for retracing the steps taken in differentiation. The inverse operation of finding an integral is, in general, much more difficult than the direct operation of finding a differential or a derivative.

This chapter gives an exposition of the fundamental rules and methods employed in integration. One or more of these rules and methods will come into play in every case in which integration is required.

18. Fundamental integrals. Following is a list of fundamental formulæ of integration derived from the fundamental formulæ of differentiation. They can be verified by differentiation, as indicated in the first of the set. An additional list is given in Art. 22.

Every integrable form* is reducible to one or more of the integrals given in these two lists. The student should have ready command of these formulæ for two reasons: first, so that he may be able to integrate these forms *immediately;* and second, so that he may know the forms *at which to aim* in reducing complicated functions. The functions to be integrated will not usually present themselves in terms of these simple, immediately integrable expressions; and therefore a considerable part of this book is taken up with algebraic and trigonometric transformations showing how to reduce given functions to these forms. In the following formulæ, u denotes any function of a single independent variable.

I. $$\int u^n \, du = \frac{u^{n+1}}{n+1} + c,$$

in which n has *any* constant value, excepting -1. The case in which $n = -1$ is given in II.

Differentiation of each member of I. with respect to u gives $u^n \, du$.

II.† $$\int \frac{du}{u} = \log u + c_0 = \log u + \log c = \log cu.$$

The different ways in which the arbitrary constant of integration can appear in this form, may be noted.

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* "An *integrable* form" here means a function whose integral can be expressed in a finite form which involves only algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

† According to I., $\int u^{-1} \, du = \frac{u^{-1+1}}{-1+1} + c = \frac{1}{0} + c = \infty + c$. Nevertheless, $\int u^{-1} \, du$ can be derived directly by means of I. For, on putting $\frac{-1}{n+1} + c$, for c, which is allowable by Art. 9, $\int u^n \, du = \frac{u^{n+1} - 1}{n+1} + c_1$. Now $\frac{u^{n+1} - 1}{n+1} = \frac{0}{0}$ when $n = -1$. Evaluation of this indeterminate form by the method of the differential calculus gives, differentiating numerator and denominator as to n, $u^{n+1} \log u$, that is, $\log u$. Hence $\int u^{-1} \, du = \log u + c_1$.

III. $$\int a^u\,du = \frac{a^u}{\log a} + c.$$

IV. $$\int e^u\,du = e^u + c.$$

V. $$\int \sin u\,du = -\cos u + c.$$

VI. $$\int \cos u\,du = \sin u + c.$$

VII. $$\int \sec^2 u\,du = \tan u + c.$$

VIII. $$\int \csc^2 u\,du = -\cot u + c.$$

IX. $$\int \sec u \tan u\,du = \sec u + c.$$

X. $$\int \csc u \cot u\,du = -\csc u + c.$$

XI. $$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = -\cos^{-1} u + c_1.$$

XII. $$\int \frac{du}{1+u^2} = \tan^{-1} u + c = -\cot^{-1} u + c_1.$$

XIII. $$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + c = -\csc^{-1} u + c_1.$$

XIV. $$\int \frac{du}{\sqrt{2u-u^2}} = \text{vers}^{-1} u + c = -\text{covers}^{-1} u + c_1.$$

Ex. 1. $$\int x^3dx = \frac{x^{3+1}}{3+1} + c = \tfrac{1}{4}x^4 + c.$$

Ex. 2. $$\int \frac{dx}{x^5} = \int x^{-5}dx = \frac{x^{-5+1}}{-5+1} + c = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c.$$

Ex. 3. $$\int \frac{\cos x\,dx}{1+\sin x} = \int \frac{d(1+\sin x)}{1+\sin x} = \log(1+\sin x) + c$$

Ex. 4. $\int x^4dx,\ \int x^{10}dx,\ \int s^{m+n}ds,\ \int t^{99}dt,\ \int z^{21}dz,\ \int p^{37}dp.$

Ex. 5. $\int_3^5 x^{-3}dx,\ \int_5^8 \frac{dx}{x^2},\ \int \frac{dx}{x^7},\ \int x^{-10}dx,\ \int x^{-m-1}dx,\ \int x^{\text{[illegible]}1}\,dx,\ \int \frac{dx}{x^{21}}.$

Ex. 6. $\int x^{\frac{2}{3}}dx,\ \int t^{\frac{1}{5}}dt,\ \int x^{\frac{3}{4}}dx,\ \int v^{\frac{p}{q}}dv,\ \int x^{-\frac{2}{7}}dx,\ \int x^{-\frac{3}{5}}dx,\ \int x^{-\frac{p}{q}}dx.$

Ex. 7. $\int \sqrt{x}\,dx,\ \int \frac{dx}{\sqrt{x}},\ \int \frac{dx}{\sqrt{x^3}},\ \int \sqrt{x^3}\,dx,\ \int \sqrt[5]{x^2}\,dx,\ \int \sqrt{x^5}\,dx,\ \int \frac{dx}{\sqrt[4]{x^3}}.$

Ex. 8. $\int \frac{dt}{t},\quad \int \frac{ds}{s-1},\ \int \frac{4x^2dx}{x^3+3},\quad \int \frac{d(uvw)}{uvw+1},\quad \int \frac{(3x^2+4x-2)\,dx}{x^3+2x^2-2x+4},$

$\int \frac{\sec^2 x\,dx}{\tan x},\quad \int \frac{\cos x\,dx}{\sin x}.$

Ex. 9. $\int e^{2x}2\,dx,\ \int_0^4 2^x\,dx,\ \int (m+n)^x\,dx.$

Ex. 10. $\int \sin 2x\,d(2x),\ \int \cos 3x\,d(3x),\ \int \sec^2 4x\,d(4x).$

Ex. 11. $\int \sec\tfrac{1}{2}x\tan\tfrac{1}{2}x\,d(\tfrac{1}{2}x),\ \int \frac{2\,dx}{\sqrt{1-4x^2}},\ \int \frac{3\,dx}{\sqrt{1-9x^2}},\ \int \frac{d(uv)}{\sqrt{1-u^2v^2}}.$

Ex. 12. $\int \frac{2\,dx}{1+4x^2},\ \int \frac{3\,dx}{1+9x^2},\ \int \frac{2\,dx}{2x\sqrt{4x^2-1}},\ \int \frac{4\,dx}{4x\sqrt{16x^2-1}},\ \int \frac{d\left(\frac{x}{a}\right)}{\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1}}.$

19. Two universal formulæ of integration. In this article two formulæ of integration will be given which differ from those of Art. 18 in that they do not apply to particular forms merely, but are of a much more general character.

Suppose that $f(x)$, $F(x)$, $\phi(x)$, $\cdots$, are any functions of x. Then,

$$\textbf{A.}\quad \int \left\{ f(x) + F(x) + \phi(x) + \cdots \right\} dx = \int f(x)dx + \int F(x)dx + \int \phi(x)dx + \cdots;$$

for differentiation of each member of A gives $f(x) + F(x) + \phi(x) + \cdots$. This formula may be thus expressed: *the integral of the sum of any number of functions is equal to the sum of the integrals of the several functions.*

Ex. 1. $$\int(x^2 \pm \sin x \pm e^x)\,dx = \int x^2\,dx \pm \int \sin x\,dx \pm \int e^x\,dx,$$
$$= \tfrac{1}{3}x^3 \mp \cos x \pm e^x + c.$$

Each of the separate integrations requires an arbitrary constant; but, since all these constants are connected by the signs + and −, their algebraic sum is equivalent to a single constant.

Again, if u is any function of x, and m is a constant,

B. $$\int mu\,dx = m\int u\,dx,$$

for differentiation of each member of **B** gives mu. Hence, a constant factor can be removed from one side of the sign of integration to the other without affecting the value of the integral. It will soon be found that sometimes it will make the work simpler to remove such a factor from the right to the left of the sign $\int$, and that, at other times, the process of integration will be aided by putting such a factor under the integration sign. It follows from *B*, that the value of an integral is unaltered if a constant is used as a multiplier on one side of the sign $\int$, and as a divisor on the other. Thus,

$$\int u\,dx = \frac{1}{m}\int mu\,dx = m\int\frac{u\,dx}{m}.$$

This principle will often be found useful.

Ex. 1. $\int 3x\,dx = 3\int x\,dx = \frac{3}{2}\int 2x\,dx = \frac{3}{2}x^2 + c.$

Note. The value of an integral is changed if an expression that contains x is transferred from one side of the sign $\int$ to the other. Thus,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + c;$$

but $$x\int x\,dx = \tfrac{1}{2}x^3 + c.$$

Ex. 2. $\int 7x^4\,dx$, $\int az^{10}\,dz$, $\int ac^2bx^{d-1}\,dx$, $\int a^2x^{\sqrt{2}}\,dx$, $\int 4ab^2cx^{2a\sqrt{b}-1}\,dx.$

Ex. 3. $\int (x^2 - 2x + 5)\,dx,\ \int (x^3 - 3x^{\frac{1}{2}} + 5x^{\frac{3}{2}})\,dx.$

Ex. 4. $\int (3 + 5t)^2\,dt,\ \int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3\,dx,\ \int (\cos\theta + \sin\theta)\,d\theta.$

Ex. 5. $\int e^{ax}\,dx,\ \int e^{2x}\,dx,\ \int e^{\frac{3}{2}x}\,dx,\ \int e^{\frac{p}{b}x}\,dx,\ \int e^{x^2+4x+3}(x+2)\,dx.$

Ex. 6. $\int (\cos mx + \sin 3x)\,dx,\ \int \{\sec^2 2x + \operatorname{cosec}^2 (m+n)x\}\,dx.$

Ex. 7. $\int \frac{x^{n-1}\,dx}{a + bx^n},\ \int \frac{v^2\,dv}{4 + 3v^3},\ \int \frac{x^3\,dx}{5 - 2x^4}.$

Ex. 8. $\int \frac{a^2 d\theta}{4\sqrt{1-\theta^2}},\ \int \frac{dx}{9 + 4x^2},\ \int \frac{dy}{16 + 25y^2}.$

20. Integration aided by a change of the independent variable. Integration can often be facilitated by a convenient change of the independent variable. For instance, if $f(x)\,dx$ is not immediately integrable, it may be possible to change the independent variable from x to t, the relation between x and t being, say $x = \psi(t)$, so that $f(x)\,dx$ is thereby put into a form $F(t)\,dt$ which can be easily integrated. Experience and practice afford the only means of determining the substitutions that will be helpful in particular cases. The actual substitution of the new variable may often be conveniently omitted, as in Exs. 1, 2, 3, below.

Ex. 1. $\int (x+a)^n\,dx.$

On putting $x + a = t$, $dx = dt$; and the given integral becomes

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + c = \frac{(x+a)^{n+1}}{n+1} + c.$$

Since $dx = d(x+a)$, the given integral may also be written

$\int (x+a)^n d(x+a)$, and $x + a$ being regarded as the variable u, the integral is $\frac{(x+a)^{n+1}}{n+1} + c$, as before.

Ex. 2. $\int \sec^2 (5 - 2x)\, dx.$

If $5 - 2x = t$, $dx = -\frac{1}{2} dt$; and

$$\int \sec^2 (5 - 2x)\, dx = -\tfrac{1}{2}\int \sec^2 t\, dt = -\tfrac{1}{2} \tan t + c$$
$$= -\tfrac{1}{2} \tan (5 - 2x) + c.$$

Since $dx = -\frac{1}{2} d(5 - 2x)$, the integral may be written also

$$\int \sec^2 (5 - 2x)\, dx = -\tfrac{1}{2}\int \sec^2 (5 - 2x) d(5 - 2x)$$
$$= -\tfrac{1}{2} \tan (5 - 2x) + c.$$

Ex. 3. $\int e^{a+bx} dx.$

If $a + bx = t$, $dx = \frac{1}{b} dt$, and

$$\int e^{a+bx} dx = \frac{1}{b}\int e^t dt = \frac{1}{b} e^t + c = \frac{1}{b} e^{a+bx} + c.$$

Otherwise: since $dx = \frac{1}{b} d(a + bx)$,

$$\int e^{a+bx} dx = \frac{1}{b}\int e^{a+bx} d(a + bx) = \frac{1}{b} e^{a+bx} + c.$$

Ex. 4. $\int \frac{12x^2 - 4x + 5}{4x^3 - 2x^2 + 5x - 10} dx.$

On putting $4x^3 - 2x^2 + 5x - 10 = t$, it follows that $(12x^2 - 4x + 5)dx = dt$, and

$$\int \frac{12x^2 - 4x + 5}{4x^3 - 2x^2 + 5x - 10} dx = \int \frac{dt}{t} = \log t + c = \log (4x^3 - 2x^2 + 5x - 10) + c.$$

Note. *If the expression under the sign of integration is a fraction whose numerator is the differential of the denominator, the integral is the logarithm of the denominator.*

Ex. 5. $\int \frac{\cos x\, dx}{\sin^6 x}.$

If $\sin x = t$, $\cos x\, dx = dt$; and

$$\int \frac{\cos x\, dx}{\sin^6 x} = \int \frac{dt}{t^6} = -\frac{1}{5t^5} + c = -\frac{1}{5 \sin^5 x} + c.$$

Otherwise: $\int \frac{\cos x\, dx}{\sin^6 x} = \int \frac{d(\sin x)}{(\sin x)^6} = -\frac{1}{5 \sin^5 x} + c.$

The necessity of learning to recognize *forms* readily will be apparent.

Ex. 6. $\int \frac{x^3\,dx}{(x+1)^4}$.

If $$x+1=z,\ dx=dz,\ x=z-1,$$

then $$\int \frac{x^3\,dx}{(x+1)^4} = \int \frac{(z-1)^3}{z^4}dz = \int \left(\frac{1}{z} - \frac{3}{z^2} + \frac{3}{z^3} - \frac{1}{z^4}\right)dz$$

$$= \log z + \frac{3}{z} - \frac{3}{2\,z^2} + \frac{1}{3\,z^3} + c = \log z + \frac{18\,z^2 - 9\,z + 2}{6\,z^3} + c$$

$$= \log(x+1) + \frac{18\,x^2 + 27\,x + 11}{6\,(x+1)^3} + c.$$

Ex. 7. $$\int \frac{e^{2x}}{(e^x+1)^{\frac{1}{4}}}\,dx.$$

If $$e^x + 1 = z,\ e^x\,dx = dz,\ e^x = z - 1;$$

and $$\int \frac{\rho^{2x}}{(e^x+1)^{\frac{1}{4}}}\,dx = \int \frac{e^x \cdot e^x\,dx}{(e^x+1)^{\frac{1}{4}}} = \int \frac{(z-1)}{z^{\frac{1}{4}}}\,dz = \int (z^{\frac{3}{4}} - z^{-\frac{1}{4}})\,dz$$

$$= \tfrac{4}{7}\,z^{\frac{7}{4}} - \tfrac{4}{3}\,z^{\frac{3}{4}} + c = \tfrac{4}{21}\,z^{\frac{3}{4}}\,(3\,z - 7)$$

$$= \tfrac{4}{21}(e^x+1)^{\frac{3}{4}}(3\,e^x - 4).$$

Ex. 8. $\int (x+a)^{\frac{3}{2}}\,dx$, $\int (x+a)^{\frac{1}{2}}\,dx$, $\int \frac{dx}{x+a}$, $\int \frac{dx}{\sqrt{x+a}}$, $\int \frac{dx}{(x+a)^2}$, $\int (2+3\,x)^{\frac{1}{5}}\,dx$, $\int (3-7\,x)^{\frac{2}{3}}\,dx$.

Ex. 9. $\int \cos(x+a)\,dx$, $\int \sec^2(x+a)\,dx$, $\int \frac{dx}{\cos^2(4-3\,x)}$, $\int \sin(a+bx)\,dx$.

Ex. 10. $\int \cos\frac{x}{2}\,dx$, $\int e^{2+5x}\,dx$, $\int e^{\frac{x}{3}}\,dx$, $\int \frac{\cos x}{\sin^3 x}\,dx$.

Ex. 11. $\int x(a+x)^{\frac{1}{3}}dx$, $\int \frac{x\,dx}{(a+bx)^{\frac{1}{3}}}$.

Ex. 12. $\int \frac{dx}{(1+x^2)\tan^{-1}x}$, $\int \frac{\cos(\log x)\,dx}{x}$, $\int \frac{d\theta}{\sin^2\left(\frac{\theta}{n}\right)}$.

Ex. 13. $\int (a+bz)^3\,dz$, $\int \sqrt[3]{(a+bx)^2}\,dx$, $\int \frac{dy}{\sqrt[7]{(a+by)^4}}$.

Ex. 14. $\int \frac{dx}{\sqrt{15 + 4x - 4x^2}}$. (Put $2x - 1 = 4z$.)

Ex. 15. $\int \frac{\cos x\, dx}{\sin^3 x}$. (Put $\sin x = z$.) $\int (\sin^4 \theta - 3 \sin^3 \theta + 4 \sin^2 \theta + 11 \sin \theta + 2) \cos \theta\, d\theta$, $\int (\tan^3 \phi - 7 \tan^2 \phi + 2 \tan \phi + 9) \sec^2 \phi\, d\phi$.

21. Integration by parts. Two universal formulæ of integration were given in Art. 19. A third formula of this kind will now be discussed. Differentiation shows that, u and v being any functions of x,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

This may be written also in the differential form,

$$d\,(uv) = v\left(\frac{du}{dx}\right)dx + u\left(\frac{dv}{dx}\right)dx,$$

or more simply,

$$d\,(uv) = v\,du + u\,dv, \tag{1}$$

in which,

$$du = \frac{du}{dx}dx, \text{ and } dv = \frac{dv}{dx}dx.$$

Equation (1) becomes on transposition,

$$u\,dv = d(uv) - v\,du.$$

Integration of both members of this equation gives

C. $$\int \boldsymbol{u\,dv = uv - \int v\,du}.$$

Equation **C** may be used as a formula for integrating $u\,dv$ when the integral of $v\,du$ can be found. This method of integration, commonly called "integration by parts," may be adopted when $f(x)\,dx$ is not immediately integrable, but can be resolved into two factors, say u and dv, such that the integrals of dv and $v\,du$ are easily obtained. The procedure is as follows:

$$\int f(x)\,dx = \int u\,dv;$$

whence by **C**,

$$= uv - \int v\,du.$$

No general rule can be given for choosing the factors u and dv. Facility in using formula **C** can be obtained only by practice. This formula has a greater importance and a wider application, than any other in the integral calculus. The following examples will show how it may be employed.

Ex. 1. Integrate $x \sin x\, dx$.

Let $\quad u = x, \quad dv = \sin x\, dx;$

then, $\quad du = dx, \quad v = -\cos x.$

Application of **C** gives

$$\int x \sin x\, dx = -x \cos x + \int \cos x\, dx;$$

$$= -x \cos x + \sin x + c.$$

Ex. 2. Integrate $\log x\, dx$.

Let $\quad u = \log x, \quad dv = dx;$

then, $\quad du = \frac{dx}{x}, \quad v = x.$

Therefore by **C**,

$$\int \log x\, dx = x \log x - \int x \frac{dx}{x}$$

$$= x \log x - x + c.$$

Ex. 3. Find $\int xe^x\, dx$.

Let $\quad u = e^x, \quad dv = x\, dx;$

then, $\quad du = e^x\, dx, \quad v = \frac{1}{2} x^2.$

The formula gives

$$\int xe^x\, dx = \tfrac{1}{2} x^2 e^x - \tfrac{1}{2} \int x^2 e^x\, dx.$$

But $x^2 e^x\, dx$ is not so simple for integration as $xe^x\, dx$. This indicates that a different choice of factors should have been made.

On putting $\quad u = x, \quad dv = e^x\, dx,$

$du = dx, \quad v = e^x,$

and formula **C** now gives

$$\int xe^x\, dx = xe^x - \int e^x\, dx$$

$$= xe^x - e^x + c.$$

Ex. 4. Find $\int x^3 e^x \, dx$.

Let $u = x^3, \ dv = e^x \, dx$; then $du = 3x^2 \, dx, \ v = e^x$.

Hence, $$\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx.$$

To find $\int x^2 e^x \, dx$, put $u = x^2, \ dv = e^x \, dx$; then $du = 2x \, dx, \ v = e^x$; and

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

By Ex. 3, $$\int x e^x \, dx = x e^x - e^x + c.$$

Hence, combining the results,

$$\int x^3 e^x \, dx = e^x (x^3 - 3x^2 + 6x - 6) + c.$$

This is an example in which several successive operations of the same kind are required in order to effect the integration. Many such examples will be met, and usually a formula called "a formula of reduction" will be found for integrating them. "Integration by parts" is of great use in deducing these formulæ of reduction. In order to avoid making mistakes in cases like Ex. 4, a good plan is to write down the successive steps in the integration clearly, without putting in the intermediate work, which can be kept in another place. Thus:

$$\begin{aligned} \int x^3 e^x \, dx &= x^3 e^x - 3 \int x^2 e^x \, dx \\ &= x^3 e^x - 3 \left[x^2 e^x - 2 \int x e^x \, dx \right] \\ &= x^3 e^x - 3 \left[x^2 e^x - 2 (x e^x - e^x) \right] + c \\ &= e^x (x^3 - 3x^2 + 6x - 6) + c. \end{aligned}$$

Ex. 5. $\int \sin^{-1} x \, dx$.

Ex. 6. $\int \cot^{-1} x \, dx$.

Ex. 7. $\int z a^z \, dz$.

Ex. 8. $\int x^2 a^x \, dx$.

Ex. 9. $\int \tan^{-1} x \, dx$.

Ex. 10. $\int x^2 \sin x \, dx$.

Ex. 11. $\int x^2 \cos x \, dx$.

Ex. 12. $\int x \tan^{-1} x \, dx$.

Ex. 13. $\int (\log x)^2 \, dx$.

Ex. 14. $\int \cos \theta \log \sin \theta \, d\theta$.

Ex. 15. $\int \sec^2 x \log \tan x \, dx.$

Ex. 16. $\int x^3 (\log x)^2 \, dx.$

Ex. 17. $\int x e^{mx} \, dx.$

Ex. 18. $\int x^m \log x \, dx.$

22. Additional standard forms. Some fundamental integrals which often appear are collected together in the following list. Their derivation will be found in the next article.

$$\textbf{XV.} \quad \int \tan u \, du = \log \sec u + C.$$

$$\textbf{XVI.} \quad \int \cot u \, du = \log \sin u + C.$$

$$\textbf{XVII.} \quad \int \sec u \, du = \log \tan \left(\frac{u}{2} + \frac{\pi}{4} \right) + C.$$

$$\textbf{XVIII.} \quad \int \operatorname{cosec} u \, du = \log \tan \frac{u}{2} + C.$$

$$\textbf{XIX.} \quad \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C = -\cos^{-1} \frac{u}{a} + C'.$$

$$\textbf{XX.} \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C = -\frac{1}{a} \cot^{-1} \frac{u}{a} + C'.$$

$$\textbf{XXI.} \quad \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{u}{a} + C'.$$

$$\textbf{XXII.} \quad \int \frac{du}{\sqrt{2au - u^2}} = \operatorname{vers}^{-1} \frac{u}{a} + C = -\operatorname{covers}^{-1} \frac{u}{a} + C'.$$

$$\textbf{XXIII.} \quad \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u - a}{u + a} + C = \frac{1}{a} \tanh^{-1} \frac{u}{a} + C'.$$

$$\textbf{XXIV.} \quad \int \frac{du}{\sqrt{u^2 + a^2}} = \log (u + \sqrt{u^2 + a^2}) + C = \sinh^{-1} \frac{u}{a} + C'.$$

$$\textbf{XXV.} \quad \int \frac{du}{\sqrt{u^2 - a^2}} = \log (u + \sqrt{u^2 - a^2}) + C = \cosh^{-1} \frac{u}{a} + C'.$$

23. Derivation of the additional standard forms.

Formula XV. Since $\tan u = \dfrac{\sin u}{\cos u}$,

$$\int \tan u\, du = \int \frac{\sin u}{\cos u}\, du = -\int \frac{d\,(\cos u)}{\cos u}$$

$$= -\log \cos u = \log \sec u.$$

Formula XVI. $\displaystyle\int \cot u\, du = \int \frac{\cos u}{\sin u}\, du = \int \frac{d\,(\sin u)}{\sin u}$

$$= \log \sin u = -\log \operatorname{cosec} u.$$

Formula XVIII. Since $\operatorname{cosec} u = \operatorname{cosec} u\, \dfrac{\operatorname{cosec} u - \cot u}{\operatorname{cosec} u - \cot u}$,

$$\int \operatorname{cosec} u\, du = \int \frac{-\operatorname{cosec} u \cot u + \operatorname{cosec}^2 u}{\operatorname{cosec} u - \cot u}\, du$$

$$= \int \frac{d\,(\operatorname{cosec} u - \cot u)}{\operatorname{cosec} u - \cot u}$$

$$= \log\,(\operatorname{cosec} u - \cot u) = \log \frac{1 - \cos u}{\sin u}$$

$$= \log \frac{2 \sin^2 \frac{u}{2}}{2 \sin \frac{u}{2} \cos \frac{u}{2}} = \log \tan \frac{u}{2}.$$

Formula XVII. On substituting $u + \frac{\pi}{2}$ for u in XVIII., there results

$$\int \operatorname{cosec}\left(u + \frac{\pi}{2}\right) du = \log \tan\left(\frac{u}{2} + \frac{\pi}{4}\right),$$

that is,

$$\int \sec u\, du = \log \tan\left(\frac{u}{2} + \frac{\pi}{4}\right).$$

Formula XIX. If $u = az$, then $du = a\,dz$, and

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a\,dz}{\sqrt{a^2 - a^2z^2}} = \int \frac{dz}{\sqrt{1 - z^2}} = \sin^{-1} z$$

$$= \sin^{-1} \frac{u}{a} = -\cos^{-1} \frac{u}{a}.$$

Formula XX. If $u = az$, then $du = a\,dz$, and

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\int \frac{dz}{1 + z^2} = \frac{1}{a}\tan^{-1} z$$

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} = -\frac{1}{a}\cot^{-1}\frac{u}{a}.$$

Formulæ XXI., XXII. These integrals also can be derived by means of the substitution $u = az$.

Formula XXIII. Since $\frac{1}{u^2 - a^2} = \frac{1}{2\,a}\left(\frac{1}{u - a} - \frac{1}{u + a}\right)$,

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2\,a}\int \left(\frac{1}{u - a} - \frac{1}{u + a}\right) du$$

$$= \frac{1}{2\,a}\{\log(u - a) - \log(u + a)\}$$

$$= \frac{1}{2\,a}\log\frac{u - a}{u + a}.$$

Formula XXIV. If $u^2 + a^2 = z^2$, then $u\,du = z\,dz$,

or $$\frac{du}{z} = \frac{dz}{u}.$$

Hence, $$\frac{du}{\sqrt{u^2 + a^2}} = \frac{du}{z} = \frac{dz}{u},$$

and, by composition, $$= \frac{du + dz}{u + z}.$$

Therefore, $$\int \frac{du}{\sqrt{u^2 + a^2}} = \int \frac{du + dz}{u + z}$$

$$= \log(u + z) + c = \log(u + \sqrt{u^2 + a^2}) + c$$

$$= \log\frac{u + \sqrt{u^2 + a^2}}{a} + c_1 = \sinh^{-1}\frac{u}{a} + c_1.$$

The latter result can be derived also in the following way:

On putting $u = az$,

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \int \frac{dz}{\sqrt{z^2 + 1}} = \sinh^{-1} z + c.$$

$$= \sinh^{-1} \frac{u}{a} + c.$$

Formula XXV. Similarly, on putting $u^2 - a^2 = z^2$,

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \log\left(u + \sqrt{u^2 - a^2}\right) + c$$

$$= \log \frac{u + \sqrt{u^2 - a^2}}{a} + c_2 = \cosh^{-1} \frac{u}{a} + c_2.$$

Or, putting $u = az$,

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \int \frac{dz}{\sqrt{z^2 - 1}} = \cosh^{-1} z + c = \cosh^{-1} \frac{u}{a} + c.$$

Ex. 1. Find $\int \sqrt{a^2 - x^2}\, dx.$

Integrating by parts, let

$$u = \sqrt{a^2 - x^2}, \quad dv = dx.$$

Then
$$du = -\frac{x}{\sqrt{a^2 - x^2}}\, dx, \quad v = x,$$

and
$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + \int \frac{x^2\, dx}{\sqrt{a^2 - x^2}}.$$

Since $\sqrt{a^2 - x^2} = \dfrac{a^2 - x^2}{\sqrt{a^2 - x^2}}$, it follows that $\dfrac{x^2}{\sqrt{a^2 - x^2}} = \dfrac{a^2}{\sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}.$

Hence, $\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \int \dfrac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2}\, dx.$

From this, on transposing the last integral to the first member,

$$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right).$$

Ex. 2. $$\int \frac{dx}{\sqrt{24+10x-x^2}} = \int \frac{dx}{\sqrt{49-(x^2-10x+25)}}$$
$$= \int \frac{dx}{\sqrt{7^2-(x-5)^2}} = \sin^{-1}\left(\frac{x-5}{7}\right).$$

Ex. 3. $$\int \frac{dx}{\sqrt{x^2+8x+52}} = \int \frac{dx}{\sqrt{(x+4)^2+36}}$$
$$= \log\left(x+4+\sqrt{x^2+8x+52}\right).$$

Ex. 4. $$\int \frac{dx}{x^2+6x+12} = \int \frac{dx}{(x+3)^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+3}{\sqrt{3}}.$$

Ex. 5. $$\int \frac{dx}{x^2+6x+5} = \int \frac{dx}{(x+3)^2-4} = \frac{1}{4}\log\frac{(x+3)-2}{(x+3)+2}$$
$$= \frac{1}{4}\log\frac{x+1}{x+5}.$$

Ex. 6. $\displaystyle\int \frac{dx}{\sqrt{x^2-5x}}.$

Ex. 7. $\displaystyle\int \frac{\sqrt{7}\,dx}{\sqrt{7x^2+19}}.$

Ex. 8. $\displaystyle\int \frac{5x\,dx}{\sqrt{3x^4+2x^2-1}}.$

Ex. 9. $\displaystyle\int \frac{dx}{-3+4x-x^2}.$

Ex. 10. $\displaystyle\int \frac{5\,dx}{\sqrt{4x-x^2}}.$

Ex. 11. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}.$

Ex. 12. $\displaystyle\int \frac{d\theta}{\sqrt{5-\theta^2}}.$

Ex. 13. $\displaystyle\int 6\tan 3x\,dx.$

Ex. 14. $\displaystyle\int \frac{7\,dx}{\sqrt{3-5x^2}}.$

Ex. 15. $\displaystyle\int \frac{dx}{a+x^2}.$

Ex. 16. $\displaystyle\int \frac{dy}{y^2-8}.$

Ex. 17. $\displaystyle\int \frac{dx}{\tan ax}.$

Ex. 18. $\displaystyle\int \cot(ax+b)dx.$

Ex. 19. $\displaystyle\int \frac{2\,dz}{4z^2+3}.$

Ex. 20. $\displaystyle\int \frac{dx}{x^2-4x+8}\left[=\int \frac{d(x-2)}{(x-2)^2+4}\right]$

Ex. 21. $\displaystyle\int \frac{dx}{x^2-4x-8}.$

Ex. 22. $\displaystyle\int (\sec 2x+1)^2\,dx.$

Ex. 23. $\displaystyle\int \frac{dx}{(x-a)\sqrt{(x-a)^2-b^2}}.$

Ex. **24.** $\int \frac{dx}{4x\sqrt{4x^2-5}}$.

Ex. **25.** $\int \frac{x\,dx}{\sqrt{x^4-c^4}}$.

Ex. **26.** $\int xy\sqrt{x^2-y^2}\,dx$.

Ex. **27.** $\int \frac{d\beta}{\sqrt{\beta^2+2\sqrt{3}}}$.

Ex. **28.** $\int \frac{dx}{\sqrt{15x-6x^2}}$.

Ex. **29.** $\int \frac{dx}{x^2+6x+13}$.

Ex. **30.** $\int \frac{1+\cos\theta}{\sin\theta}\,d\theta$.

Ex. **31.** $\int \frac{dx}{\sqrt{a^2x^2+2bx+c}}$.

Ex. **32.** $\int \sqrt{\frac{x+1}{x-1}}\,dx$.

[Rationalize the numerator.]

24. Integration of a total differential. It has been shown in the differential calculus, that if

$$u = f(x, y), \tag{1}$$

x, y, being independent variables, the total differential of u is equal to the sum of its partial differentials with respect to x and y. That is,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \tag{2}$$

It will be remembered that when differentiation is performed with respect to x, y is regarded as constant, and when differentiation is performed with respect to y, x is regarded as constant.

Suppose that a differential with respect to two independent variables is given, namely,

$$P\,dx + Q\,dy, \tag{3}$$

in which P and Q are functions of x and y. The anti-differential of (3) is required. Not every function (3) that may be written at random has an anti-differential. Hence, it is necessary to determine whether an anti-differential of (3) exists or not, before trying to find it. It has been shown in the differential calculus that if u and its first and second partial derivatives with regard to x, y are continuous functions of x, y, then,

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}. \tag{4}$$

If (3) has an integral, say u, then,

$$du = P\,dx + Q\,dy, \tag{5}$$

in which
$$P = \frac{\partial u}{\partial x}, \tag{6}$$

and
$$Q = \frac{\partial u}{\partial y}. \tag{7}$$

Differentiation of both members of (6) and (7) with respect to y and x, respectively, gives

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Hence, by (4),
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{8}$$

Therefore, if (3) has an integral, relation (8) holds between the coefficients P, Q, and the differential (3) is then said to be *an exact differential.* Conversely, it can be shown that if relation (8) holds, the differential (3) has an integral. For the present the latter proposition may be assumed to be true.* The condition (8) is called the criterion of the integrability of the differential (3).

Suppose that the coefficients P, Q satisfy the test (8), then there is a function u which satisfies equation (5). Since $P\,dx$ can have been derived only from the terms that contain x, integration of the second member of (5) with respect to x gives

$$\int P\,dx + c,$$

in which c denotes any expression not involving x.

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* For proof, see *Introductory Course in Differential Equations*, Art. 12, by D. A. Murray (Longmans, Green & Co.).

Now $Q\,dy$ has been derived from all the terms of u that contain y. Some of these terms may contain x also; and if so, they have been discovered already in $\int P\,dx$. Therefore, the remaining terms of u that do not contain x will be found by integrating with respect to y the terms of $Q\,dy$ that do not contain x. Hence, the following rule: Integrate $P\,dx$ as if y were constant; integrate, as if x were constant, the terms in $Q\,dy$ that do not contain x; add the results and the arbitrary constant of integration.

Ex. **1.** Integrate $y\,dx + x\,dy$.

Here $P = y$, $Q = x$; hence $\frac{\partial P}{\partial y} = 1$, $\frac{\partial Q}{\partial x} = 1$, and thus, criterion (8) is satisfied.

Also, $\int P\,dx = \int y\,dx = xy$; and there are not any terms in $Q\,dy$ without x. Hence the integral is $xy + c$.

Ex. **2.** Integrate $y\,dx - x\,dy$.

Here $P = y$, $Q = -x$; hence $\frac{\partial P}{\partial y} = 1$, $\frac{\partial Q}{\partial x} = -1$, and the criterion is not satisfied. Therefore an integral of the given expression does not exist.

Ex. **3.** Integrate $(x^2 - 4\,xy - 2\,y^2)\,dx + (y^2 - 4\,xy - 2\,x^2)\,dy$.

Ex. **4.** Integrate $(a^2 - 2\,xy - y^2)\,dx - (x + y)^2\,dy$.

Ex. **5.** Integrate $(2\,ax + by + g)\,dx + (2\,ay + bx + e)\,dy$.

25. Summary. The directions so far given for obtaining the indefinite integral of $f(x)\,dx$ may be summarized as follows:

(1) Memorize the fundamental formulæ of integration given in Arts. 18 and 22.

(2) Acquire familiarity with the application of the principle of substitution, or change of the independent variable, discussed in Art. 20.

(3) Use the first and second universal formulæ of integration, *A*, *B*, given in Art. 19.

(4) Learn to apply with ease the third universal formula of integration, namely, the formula for integration by parts given in Art. 21.

EXAMPLES ON CHAPTER III

1. $\int ac^2 x^{c-1}\,dx$, $\int (m+n)\,x^{m+n-1}\,dx$, $\int (m+n)\,v^{m+n+1}\,dv$,

$\int \frac{a}{3} x^{\frac{3}{a}-1}\,dx$, $\int a^{\frac{1}{2}} b^{\frac{2}{3}} z^{\sqrt{ab}-1}\,dz$.

2. $\int \left(x^2 + \frac{2}{x^{\frac{1}{2}}} - 3\,x^{\frac{2}{3}} + \frac{5}{x^{\frac{2}{3}}}\right) dx$, $\int_1^8 (a^{\frac{2}{3}} - y^{\frac{2}{3}})^2\,dy$.

3. $\int \frac{x^4+9}{x+2}\,dx$, $\int \frac{z^3 + 2\,z - 7}{z-2}\,dz$, $\int_3^7 \frac{v^2+5}{v-1}\,dv$.

4. Find the functions whose differential coefficients are

$$x^{\frac{1}{n}},\quad x^{-\frac{1}{n}},\quad x^{-n},\quad v^{\frac{2}{3}} - 3\,av^{\frac{1}{3}} + 4\,av^{-\frac{1}{3}}.$$

5. Find the anti-differentials of

$$(\sec^2\theta + \operatorname{cosec}^2\theta)\,d\theta,\quad (3\cos 2\,\phi - 5\sin 3\,\phi)\,d\phi,\quad \frac{\sin\psi\,d\psi}{a + b\cos\psi}.$$

6. Find the anti-derivatives of

$$\left(\frac{1}{y-a} + \frac{1}{y+a}\right),\quad \cos x + \frac{1}{\cos^2 x},\quad \frac{1}{x}\log x,\quad \frac{1}{ax+b}\log(ax+b).$$

7. Evaluate the following definite integrals:

$$\int_1^4 (x^{-\frac{3}{2}} + 5\,x^2)\,dx,\quad \int_0^{\frac{\pi}{8}} \cos 4\,\theta\,d\theta,\quad \int_0^{\frac{\pi}{8}} \cos 4\,x\,dx,\quad \int_1^e \frac{4\,dz}{z},$$

$$\int_0^{\frac{1}{2}} \frac{2\,dx}{\sqrt{1-4\,x^2}},\quad 4\int_0^{\frac{\pi}{8}} \tan 2\,\theta\,d\theta.$$

8. $\log 25 \int_{-\infty}^0 5^x\,dx$, $\int_{\sqrt{2}}^{\sqrt{1+e}} \frac{x\,dx}{x^2-1}$, $\int_{-\infty}^0 (e^{ax} + e^{\frac{x}{a}})\,dx$, $\int_0^{\frac{1}{2}} (e^x + e^{-x})\,dx$

9. $\int_0^\infty e^{-2x^2+5}\,x\,dx$, $\int_{\frac{1}{\sqrt{2}}}^1 \frac{dx}{\sqrt{1-x^2}\sin^{-1}x}$, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta\log\sin\theta\,d\theta$,

$\int_0^1 e^{\sin^{-1}x} \frac{dx}{\sqrt{1-x^2}}$, $\int_1^{\sqrt{3}} e^{a\tan^{-1}x} \frac{dx}{1+x^2}$.

10. $\int_a^{3a}\left(\frac{1}{\sqrt{x+a}}+\frac{1}{\sqrt{x-a}}\right)dx,\quad \int_{\frac{\pi}{12}}^{\frac{\pi}{6}}(\sin 2\phi+\cos 3\phi)\,d\phi,$

$$\int_a^{2a}\frac{d}{dx}\left[\frac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}\right]dx.$$

11. $\int\frac{dx}{x\sqrt{x^2-a^2}}$. (Put $x=\frac{a}{z}$. Compare result with formula XXI.)

$\int\frac{dx}{\cos x}$. (Put $\sin x=z$. Compare with XVII.)

12. $\int\frac{dx}{\sqrt{1-3x-x^2}}$. (Put $3+2x=z$.) $\int\frac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}$. (Put $1+x=z^2$.)

13. $\int\frac{x^{\frac{1}{2}}-x^{\frac{2}{3}}}{2x^{\frac{1}{6}}}\,dx.\quad \int\frac{x^3dx}{\sqrt{a+bx^2}}$. (Put $a+bx^2=z^2$.)

14. $\int\sec^{-1}x\,dx,\quad \int\operatorname{cosec}^{-1}x\,dx,\quad \int\cos^{-1}x\,dx,\quad \int x^2\sin^{-1}x\,dx.$

15. $\int x\cos x\,dx,\quad \int x^3\sin x\,dx,\quad \int x\tan^2 x\,dx.$

16. $\int x^3a^x\,dx,\quad \int\sin x\log\cos x\,dx,\quad \int\operatorname{cosec}^2 x\log\cot x\,dx.$

17. $\int(2x-2)(x^2-2x+5)\cos(x^2-2x+5)\,dx$. (Put $x^2-2x+5=z$.)

18. $\int x^5\log(x^3+a^3)\,dx$. (Put $x^3+a^3=z$.)

19. $\int\frac{(\log x)^2}{x^{\frac{5}{2}}}\,dx$. (Integrate by parts, putting $u=(\log x)^2$.)

20. Show that

$$\int(\log x)^m dx=x[(\log x)^m-m(\log x)^{m-1}+m(m-1)(\log x)^{m-2}-\cdots$$
$$+(-1)^{m-1}m(m-1)\cdots3\cdot2\log x+(-1)^m\cdot m!].$$

21. Show that

$$\int e^xx^m\,dx=x^me^x-m\int e^xx^{m-1}\,dx=e^x[x^m-mx^{m-1}+m(m-1)x^{m-2}-\cdots$$
$$+(-1)^{m-1}m(m-1)\cdots3\cdot2\cdot x+(-1)^m\cdot m!].$$

22. $\int (\sec x + \operatorname{cosec} x)^2\, dx.$

23. $\int \frac{d\theta}{1 + \cot\theta}.$

24. $\int \frac{dx}{\sqrt{a^2 + b^2 - x^2}}.$

25. Evaluate $2\int_{\frac{\pi}{4}}^{\theta} \log\tan\theta \cdot \operatorname{cosec} 2\,\theta\, d\theta + \log\cot\theta \int_{\frac{\pi}{4}}^{\theta} \operatorname{cosec} 2\,\theta\, d\theta.$

26. $\int (a\tan\theta\sec\theta + b)\cot\theta\, d\theta.$

27. $\int \frac{dx}{\sqrt{2 + 2\,x - x^2}}, \qquad \int \frac{dx}{\sqrt{-16 - 10\,x - x^2}}.$

28. $\int \log\left(x + \sqrt{x^2 - a^2}\right) \frac{dx}{x^2}.$

29. $\int \log\left(x + \sqrt{x^2 + a^2}\right) \frac{dx}{\sqrt{x^2 + a^2}}.$

30. $\int e^{\int \sec\theta\, d\theta} d\theta.$

31. $\int \frac{a\sec^2\theta + b\operatorname{cosec}^2\theta}{\tan\theta + \cot\theta}\, d\theta.$

32. $\int \operatorname{vers}^{-1}\frac{x}{a} \cdot \sqrt{x}\, dx.$

33. $\int \sqrt{x^2 \pm a^2}\, dx.$

34. $\int \frac{dx}{x^{\frac{3}{2}} - 5\,x^{\frac{1}{2}}}.$ (Put $x = z^2$.)

35. $\int \frac{dx}{\sin x + \cos x}.$

36. $\int \frac{d\phi}{\cos^2\phi - \sin^2\phi}.$

37. $\int \frac{(3\,x + 4)\sqrt{x + 2}}{\sqrt{2 - x}.}\, dx.$

38. $\int \frac{\sec x \tan x\, dx}{\tan^2 x - 2}.$

39. $\int \frac{dx}{e^x - 5\,e^{-x}}.$

40. $\int \frac{dx}{\sqrt{a^{2x} - b}}.$

41. $\int \frac{\sin x\, dx}{\sin\frac{x}{2}\sqrt{\cos x}}.$

42. $\int \frac{dx}{ax^2 + bx + c}.$

43. $\int \frac{dx}{\sqrt{ax^2 + bx + c}}.$

44. $\int \frac{dx}{\sqrt{-ax^2 + bx + c}}.$

45. Integrate $\sin x \cos y\, dx + \cos x \sin y\, dy.$

46. Integrate $\cos x \cos y\, dx - \sin x \sin y\, dy.$

47. Integrate $(3\,x^2 + 6\,xy + 4\,y^2)\, dx + (3\,x^2 + 8\,xy + 6\,y^2)\, dy.$

CHAPTER IV

GEOMETRICAL APPLICATIONS OF THE CALCULUS

26. Applications of the calculus. In this chapter some of the practical applications of the integral calculus are discussed. In particular, the areas of curves and the volumes of solids of revolution are determined. Art. 32 deals with the deduction of the equation of curves from data whose expression requires the use of differential coefficients.

There is one common aim in by far the larger number of the simpler applications of the integral calculus. This aim is to find the sum of an infinite number of infinitely small quantities. The process of summation has been discussed in Chapter I. The student will find that in most of the problems there are two steps to be taken in order to obtain the solutions, viz.:

(*a*) To find the expression for any one of the infinitesimal quantities concerned and to reduce it to a form that involves only a single variable;

(*b*) To integrate this differential expression between certain limits which are assigned or are determinable.

Each of the differential expressions is called *an element*, — an element of area, an element of length, an element of volume, an element of force, etc., as the case may be.

27. Areas of curves, rectangular coördinates. It has been shown in Arts. 3–5 that the area* between the curve $y = f(x)$, the x-axis, and two ordinates for which $x=a$, $x=b$, is expressed by

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* The calculation of such an area is called "Quadrature of curves." From this comes the phrase "to perform the quadrature," which is often used as synonymous with "to integrate." The areas of only a few curves could be found before the discovery of the calculus. Giles Persone de Roberval (1602–

$$\int_a^b f(x)\,dx.$$

It has also been shown that this area can be evaluated by finding the indefinite integral of $f(x)\,dx$, substituting b and a in turn for x in the indefinite integral, and taking the difference between the results of the two substitutions.

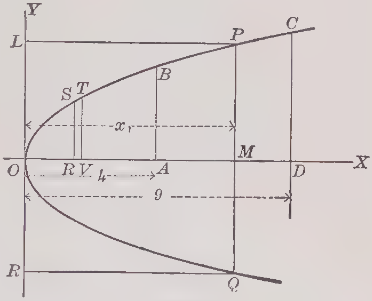

Fig. 15.

Ex. 1. Find the area bounded by the parabola whose equation is $y^2 = 4ax$, the axis of x, and the ordinate at $x = x_1$. Also find the area between the parabola $y^2 = 9x$, the axis of x, and the ordinates for which $x = 4$, $x = 9$.

Let QOC be the parabola whose equation is $y^2 = 4ax$. Take $OM = x_1$, $OA = 4$, $OD = 9$; erect the ordinates MP, AB, DC. Suppose that two ordinates RS, VT are drawn at a distance dx apart. The element of area, which is the area of any infinitesimal rectangle like $RSTV$, is $y\,dx$. The area required in the first case is equal to the sum of the areas of all such rectangles, infinite in number, that are between OY and PM; that is, between the limits zero and x_1 for x. Hence,

$$\text{area of } OPM = \int_0^{x_1} y\,dx.$$

First of all, y must be expressed in terms of x. This can be done by means of the equation of the curve, from which

$$y = \pm 2a^{\frac{1}{2}}x^{\frac{1}{2}}.$$

1675), professor of mathematics at the College of France in Paris, Blaise Pascal (1623–1662), John Wallis (1616–1703), Savilian professor of geometry at Oxford, considered an area to be made up of infinitely small rectangles, and applied the principle to the determination of the areas of parabolic curves. The French geometers found the formula for the area between the curve $y = x^m$, the axis of x, and any ordinate $x = h$ when m is a positive integer. Wallis found the area when m is negative or fractional. This was before the development of the calculus by Leibniz and Newton.

(The positive sign denotes an ordinate above the x-axis, the negative, one below.)

Hence,
$$\text{area of } OPM = \int_0^{x_1} 2\, a^{\frac{1}{2}} x^{\frac{1}{2}}\, dx$$
$$= 2\, a^{\frac{1}{2}} \left[\tfrac{2}{3} x^{\frac{3}{2}} + c\right]_0^{x_1}$$
$$= \tfrac{4}{3}\, a^{\frac{1}{2}} x_1^{\frac{3}{2}} = \tfrac{2}{3} x_1 y_1;$$

that is, area of OPM = two thirds of the area of the circumscribing rectangle $OLPM$.

The area $OPQ = 2\, OPM$ = two thirds the area of the rectangle $LPQR$.

In the second case:
$$\text{area } ABCD = \int_4^9 y\, dx$$
$$= 3\int_4^9 x^{\frac{1}{2}} dx = 3\left[\tfrac{2}{3} x^{\frac{3}{2}} + c\right]_4^9 = 38.$$

If the unit of length is an inch, the area of $ABCD$ is 38 square inches.

Ex. 2. Find the area between the curve $y^2 = 4\, ax$, the axis of y, and the line whose equation is $y = b$.

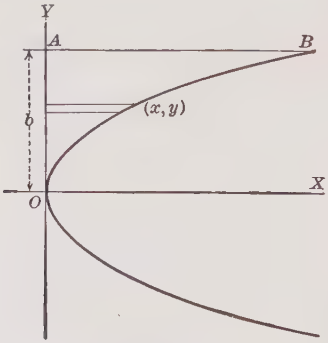

Fig. 16.

In this case it is more convenient to take for the element of area the infinitesimal rectangle indicated in the figure. The element of area is thus $x\, dy$; and
$$\text{area } OAB = \int_0^b x\, dy$$
$$= \int_0^b \frac{y^2}{4\, a} dy$$
$$= \frac{b^3}{12\, a}.$$

Ex. 3. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The area required is four times the area of the quadrant AOB. An element of area is the area of an infinitesimal rectangle $RSTV$, namely $y\,dx$. The sum of all these elements from O to A is expressed by $\int_0^a y\,dx$. From the given equation,

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2},$$

in which the positive sign denotes an ordinate above the x-axis, and the negative sign, an ordinate below. Hence,

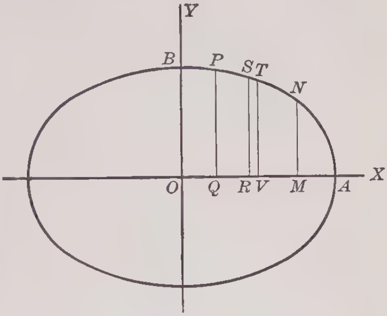

FIG. 17.

$$\text{area of ellipse} = 4\,OAB = 4\int_0^a y\,dx$$

$$=4\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx;$$

which by Ex. 1, Art. 23,

$$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]_0^a$$

$$=\pi ab.$$

If $a=b$, the ellipse is a circle whose area is πa^2.

Find the area included between the ordinates for which $x=1$, $x=4$, the curve, and the axis of x.

$$\text{Area } PQMN=\int_1^4 y\,dx=\frac{b}{a}\int_1^4\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]_1^4$$

$$=\frac{b}{a}\left\{2\sqrt{a^2-16}-\tfrac{1}{2}\sqrt{a^2-1}+\frac{a^2}{2}\left(\sin^{-1}\frac{4}{a}-\sin^{-1}\frac{1}{a}\right)\right\}.$$

If the semi-axes are 5 and 3,

$$\text{area } PQMN=\tfrac{3}{5}\{6-\sqrt{6}+\tfrac{25}{2}(\sin^{-1}\tfrac{4}{5}-\sin^{-1}\tfrac{1}{5})\}$$

$$=\tfrac{3}{5}\{6-2.454)+\tfrac{25}{2}(.927-.201)$$

$$=3.778.$$

Ex. 4. Find the area between the curve whose equation is

$$y = \tfrac{1}{12}(x-1)(x-3)(x-5),$$

the axis of x, and the ordinates for which $x = -2$, $x = 7$.

The area required $= \int_{-2}^{7} y\,dx$

$$= \tfrac{1}{12}\int_{-2}^{7}(x^3 - 9x^2 + 23x - 15)dx$$

$$= \tfrac{1}{12}\left[\tfrac{1}{4}x^4 - 3x^3 + \tfrac{23}{2}x^2 - 15x + c\right]_{-2}^{7}$$

$$= -\tfrac{99}{16}.$$

Further remark on this example may be instructive. On putting $y = 0$, the intersections of the curve and the x-axis are seen to be at the points for which $x = 1,\ 3,\ 5$. That is, referring to the figure, $OC = 1$, $OD = 3$, $OE = 5$.

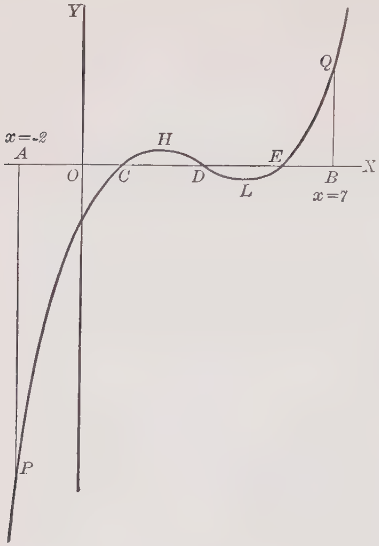

FIG. 18.

$$\text{Area } APC = \int_{-2}^{+1} y\,dx = -\tfrac{147}{16}.$$

This area appears with a negative sign, since the ordinates are negative in APC because it is below the x-axis.

$$\text{Area } CHD = \int_{1}^{3} y\,dx = +\tfrac{1}{3},$$

the sign coming out positive since CHD is above the x-axis.

$$\text{Area } DLE = \int_{3}^{5} y\,dx = -\tfrac{1}{3};$$

and $$\text{area } EQB = \int_{5}^{7} y\,dx = +3.$$

The area required = area APC + area CHD + area DLE + area EQB

$$= -\tfrac{147}{16} + \tfrac{1}{3} - \tfrac{1}{3} + 3$$

$$= -\tfrac{99}{16},$$ as obtained before.

The absolute area $= \tfrac{147}{16} + \tfrac{1}{3} + \tfrac{1}{3} + 3 = 12\tfrac{41}{48}$.

This is an example of the principle indicated in Art. 5, namely, that when the area between a curve, the x-axis, and any two ordinates, is found by integration, this area is really the sum of component areas, those above the x-axis being affected with the positive sign, and those below the x-axis with the negative sign. The next example will also serve to illustrate this.

Ex. 5. Find the area between a semi-undulation of the curve $y = \sin x$ and the x-axis.

The curve crosses the x-axis at $x = 0$, $x = \pi$, $x = 2\pi$, etc.

$$\text{Area of } ABC = \int_0^{\pi} y\,dx = \int_0^{\pi} \sin x\,dx = [-\cos x + c]_0^{\pi} = 2.$$

But $$\text{area } ABCDE = \int_0^{2\pi} y\,dx = \int_0^{2\pi} \sin x\,dx = 0.$$

The total area, regardless of sign, is 4.

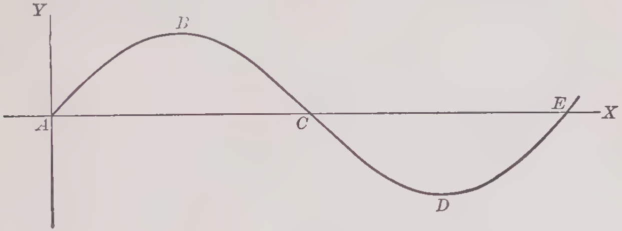

Fig. 19.

28. Precautions to be taken in finding areas by integration. The method of finding areas which has been described in the last article can be used immediately and with full confidence in the case of a curve $y = f(x)$, only when the limits a and b are finite, and the function $f(x)$ is continuous and one-valued for values of x between a and b, and does not become infinite for any value of x between a and b. Special care must be taken in cases in which any one of the conditions just mentioned does not hold. While, in some of these cases, the application of the method of Art. 27 will give true results, in other cases it will give results that are altogether erroneous. A few examples are given below in order to emphasize the necessity of caution.

Ex. 1. There is a double value for y in the parabola $y^2 = 4ax$. This was considered in Ex. 1, Art. 27.

Ex. 2. Find the area included between the parabola $(y - x - 5)^2 = x$, the axes of coördinates, and the line $x = 5$.

In this case $y = x \pm \sqrt{x} + 5$; and thus to each value of x belongs two values of y. The ambiguity can be removed by defining more exactly what area is meant. If the area $ORPM$ is desired, the value of y corresponding to each value of x between 0 and 5 is $x - \sqrt{x} + 5$. Hence,

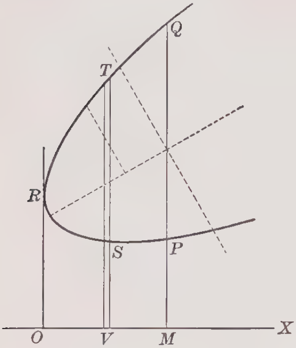

Fig. 20.

$$\text{area } ORPM = \int_0^5 (x - \sqrt{x} + 5)\,dx$$
$$= \left[\frac{x^2}{2} - \frac{2}{3}x^{\frac{3}{2}} + 5x + c\right]_0^5$$
$$= \tfrac{75}{2} - \tfrac{10}{3}\sqrt{5}.$$

If the area $ORQM$ is desired, the value of y corresponding to each value of x between 0 and 5 is $x + \sqrt{x} + 5$; and hence,

$$\text{area } ORQM = \int_0^5 (x + \sqrt{x} + 5)\,dx$$
$$= \tfrac{75}{2} + \tfrac{10}{3}\sqrt{5}.$$

If the area PRQ, between the curve and the line $x = 5$, had been required, it would have been necessary first to determine the areas $ORPM$, $ORQM$.

$$\text{Area } PRQ = \text{area } ORQM - \text{area } ORPM;$$
$$= \tfrac{20}{3}\sqrt{5}.$$

Another way of finding the area of PRQ is the following. Let TS be any infinitesimal strip of width dx parallel to the y-axis. Evidently, TS is the difference of the values of y that correspond to $x = OV$. Hence, denoting these values of y by y_1, y_2,

$$\text{area } PRQ = \int_0^5 (y_1 - y_2)\,dx$$
$$= \int_0^5 \{(x + \sqrt{x} + 5) - (x - \sqrt{x} + 5)\}\,dx$$
$$= \int_0^5 2\sqrt{x}\,dx$$
$$= \tfrac{20}{3}\sqrt{5}.$$

Ex. **3**. Find the area included between the witch $y = \dfrac{a^3}{x^2 + a^2}$ and its asymptote. The asymptote is the axis of x, and hence, the limits of integration are $+\infty$ and $-\infty$. In this case it is allowable to use infinite limits. For, on finding the area $OPQM$ between the curve, the axes, and an ordinate at distance x from the origin,

$$\text{area } OPQM = \int_0^x y\,dx$$

$$= \int_0^x \frac{a^3\,dx}{x^2 + a^2}$$

$$= \left[a^2 \tan^{-1}\frac{x}{a} + c\right]_0^x$$

$$= a^2 \tan^{-1}\frac{x}{a}.$$

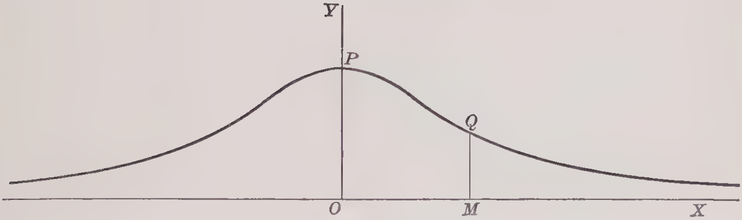

FIG. 21.

If the ordinate MQ be made to move away from the origin towards the right, that is, if the upper limit x increases continuously, then $\tan^{-1}\frac{x}{a}$ increases continuously, and approaches $\frac{\pi}{2}$ as a limit. Hence,

$$\int_0^\infty \frac{a^3dx}{x^2 + a^2} = \frac{\pi a^2}{2}$$

represents the true value of the area to the right of the y-axis. Since the curve is symmetrical with respect to the y-axis, the area required is double this, namely, πa^2.

Ex. 4. Find the area included between the curve $y^3(x^2 - a^2)^2 = 8x^3$, the x-axis, and the asymptote $x = a$.

In this case $y = \frac{2x}{(x^2 - a^2)^{\frac{2}{3}}}$. To every value of x corresponds a real value of y; but, when $x = a$, y is infinite. Therefore a special examination is required. For values of x less than a, however, y is finite. Then, for $x < a$,

$$\text{area } OMP = \int_0^x \frac{2x\,dx}{(x^2 - a^2)^{\frac{2}{3}}}$$

$$= \left[3(x^2 - a^2)^{\frac{1}{3}} + c\right]_0^x$$

$$= 3(x^2 - a^2)^{\frac{1}{3}} + 3a^{\frac{2}{3}}.$$

As x approaches the value a, it is apparent that the area OMP approaches $3\,a^{\frac{2}{3}}$ as a limit; and hence,

$$\int_0^a \frac{2\,x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 3\,a^{\frac{2}{3}}.$$

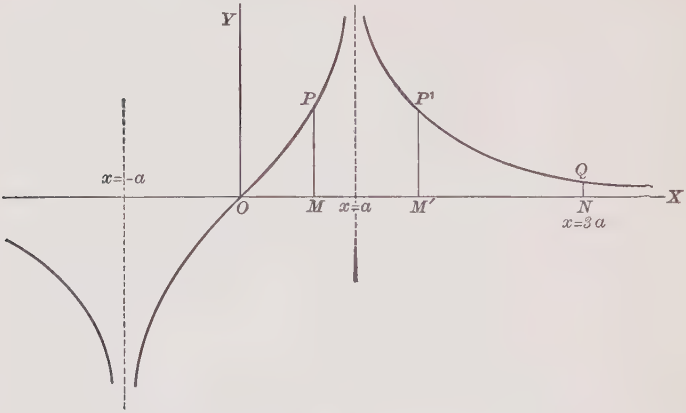

FIG. 22.

Ex. 5. Find the area bounded by the curve in Ex. 4, the x-axis, and the ordinate at $x = 3\,a$,

It has already been noticed in Ex. 4, that $f(x)$ becomes infinite when $x = a$. As a lies between the limits of integration, 0 and $3\,a$, the integration formula for the area should not be used until its applicability is determined by a special investigation. The area from $x = 0$ to the infinite ordinate at $x = a$, has been shown in Ex. 4 to be $3\,a^{\frac{2}{3}}$. The area to the right of the ordinate at $x = a$ will now be discussed. Since $f(x)$ is finite for values of x greater than a, then for limits, $x > a$ and $x = 3\,a$,

$$\begin{aligned} \text{area } M'P'QN &= \int_{x>a}^{3a} \frac{2\,x\,dx}{(x^2-a^2)^{\frac{2}{3}}} \\ &= 6\,a^{\frac{2}{3}} - 3\,(x^2-a^2)^{\frac{1}{3}}. \end{aligned}$$

As x diminishes and approaches a, this area approaches $6\,a^{\frac{2}{3}}$; and hence, the area between the infinite ordinate at $x = a$, and the ordinate at $x = 3\,a$, is $6\,a^{\frac{2}{3}}$. Hence, the total area between the curve, the x-axis, and the ordinate at $x = 3\,a$, is $3\,a^{\frac{2}{3}} + 6\,a^{\frac{2}{3}}$, that is $9\,a^{\frac{2}{3}}$.

The same result is obtained when $\int_0^{3a} \frac{2\,x\,dx}{(x^2-a^2)^{\frac{2}{3}}}$ is evaluated in the ordinary way; and thus, the integration formula for the area holds good in this case, although $f(x)$ becomes infinite for a value of x between the limits of integration.

Ex. 6. Find the area included between the curve $y(x-a)^2 = 1$, the axes, and the ordinate $x = 2a$.

Immediate application of the integration formula gives for the area,

$$\int_0^{2a} \frac{dx}{(x-a)^2} = \left[-\frac{1}{x-a} + c\right]_0^{2a} = -\frac{2}{a}.$$

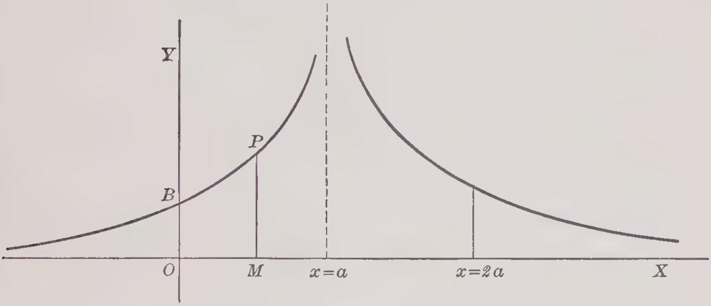

FIG. 23.

But, $f(x)$, which is the length of the ordinate y, becomes infinite for $x = a$; and, if an investigation be made similar to that carried out in Exs. 4, 5, it will be found that the area is infinite. For, OM being equal to x,

$$\text{area } OMPB = \int_0^x \frac{dx}{(x-a)^2} = \frac{1}{a-x} - \frac{1}{a}.$$

It is evident, that as x increases from 0 to a, the area increases from 0 to ∞. Consequently, the area between the curve, the axes, and the ordinate at $x = a$ is infinite. Similarly, it may be shown that the area between the curve, the x-axis, and the ordinates at $x = a$, $x = 2a$, is infinite. Therefore the total area required is infinite. Hence, the integration formula for the area, namely, $\int_0^{2a} \frac{dx}{(x-a)^2}$, fails in this particular case in which $f(x)$ becomes infinite for a value of x between the limits of integration. This conclusion may be compared with that in Ex. 5.

29. Precautions to be taken in evaluating definite integrals. It has been shown in Art. 6, that any definite integral, say $\int_a^b f(x)dx$, may be graphically represented by the area between the curve $y = f(x)$, the axis of x, and the ordinates at $x = a$, $x = b$. Hence, the statement at the beginning of Art. 28, and the precautions described in that article, must be applied when any definite integral $\int_a^b f(x)dx$ is under consideration.

EXAMPLES IN AREAS.*

1. Find by integration the areas of the triangles bounded by the coördinate axes, and each of the following lines:

$$(a)\ 7x + 5y - 35 = 0;\ (b)\ 18x - y - 12 = 0.$$

2. Find the areas of the triangles bounded by the x-axis, and

(a) the lines $7x - 3y - 21 = 0$, $x = -5$;
(b) the lines $5x + 6y + 15 = 0$, $x = -1$.

3. Find the areas of the triangles bounded by the y-axis, and

(a) the lines $9x + 4y - 6 = 0$, $y = 1$;
(b) the lines $2x + y + 8 = 0$, $y = -4$.

4. Find the area of the figure bounded by the axis of abscissas, the curve $y = x^2 + x + 1$, and the ordinates corresponding to the abscissas 2, and 3.

5. So for the curve $y = x^4 + 4x^3 + 2x^2 + 3$ between the abscissas 1, 2.

6. Find the area of the figure cut off from the curve $y = (x + 1)(x + 2)$ by the x-axis.

7. Find the area included between the semi-cubical parabola $y^2 = x^3$ and the line $x = 4$.

8. Find the area included between the semi-cubical parabola $y^2 = x^3$, the y-axis, and the line $y = 4$.

9. Find the area included by the parabola $y^2 = -4x$, and the line $x = -1$.

10. Find the area included by the parabola $x^2 + 12y = 0$, and the line $y = -3$.

11. Find the total area included by the curve $y = x^3$, and the line $y = 2x$.

12. Find the area of the first quadrant of the circle $x^2 + y^2 = r^2$.

13. Find the area intercepted between the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

14. Find the area included between the hyperbola $xy = k^2$, the x-axis, and the ordinates at $x = a$, $x = b$.

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* Figures of some of the curves referred to in examples throughout the book are given in the Appendix.

30. Volumes of solids of revolution. Let PQ be an arc of a curve whose equation is $y=f(x)$. Draw the ordinates AP, BQ, and let $OA=a$, $OB=b$. The volume of the solid $PQML$ generated by the revolution of $APQB$ about the x-axis is required. On the revolution of PQ each point in the arc PQ will describe a circle. Suppose that AB is divided into n equal parts Δx, and let $OQ_1=x$, $Q_1Q_2=\Delta x$. Construct the rectangles P_1Q_2, P_2Q_1 as indicated in the figure, and suppose that they have revolved about OX with $APQB$.

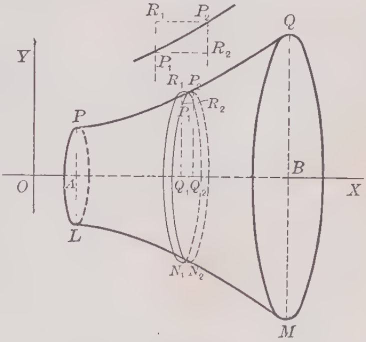

FIG. 24.

It is evident that the volume of each plate, such as $P_1P_2N_2N_1$, of the solid of revolution is less than the volume of the corresponding exterior cylinder generated by the revolution of the rectangle $Q_1R_1P_2Q_2$ about the x-axis, and greater than the volume of the corresponding interior cylinder generated by the revolution of the rectangle $Q_1P_1R_2Q_2$ about the x-axis. Now,

the volume of the cylinder generated by $Q_1P_1R_2Q_2$

$$\begin{aligned}&=\pi\overline{P_1Q_1}^2\,\Delta x\\&=\pi y^2\,\Delta x\\&=\pi\,[f(x)]^2\,\Delta x;\end{aligned}$$

and the volume of the cylinder generated by $Q_1R_1P_2Q_2$

$$\begin{aligned}&=\pi\overline{P_2Q_2}^2\,\Delta x\\&=\pi\,[f(x+\Delta x)]^2\,\Delta x.\end{aligned}$$

Hence, $$\pi\,[f(x)]^2\Delta x<P_1P_2N_2N_1<\pi\,[f(x+\Delta x)]^2\,\Delta x.$$

Suppose that $PQML$ is divided into n plates, such as $P_1P_2N_2N_1$, one plate corresponding to each segment Δx of AB; and suppose also that the interior and exterior cylinders corresponding to each of these plates are constructed. Then, on taking the sum of all the interior cylinders, and the sum of all the exterior cylinders, and the sum of all the plates $P_1P_2N_2N_1$, the latter sum being the volume required,

$$\sum_{x=a}^{x=b} \pi\,[f(x)]^2 \Delta x < PQML < \sum_{x=a}^{x=b} \pi\,[f(x+\Delta x)]^2 \Delta x.$$

As Δx approaches zero, the sum of the exterior cylinders approaches equality to the sum of the interior cylinders. The difference between these sums is at the most an infinitesimal of the first order when Δx is an infinitesimal, and accordingly has zero as its limit. Therefore, since the volume required always lies between these sums,

$$\text{volume } PQML = \operatorname*{limit}_{\Delta x \doteq 0} \sum_{x=a}^{x=b} \pi\,[f(x)]^2 \Delta x;$$

that is, $$\textbf{volume } \boldsymbol{PQML} = \int_a^b \pi\,[\boldsymbol{f(x)}]^2\, \boldsymbol{dx}.$$

The element of volume is $\pi\,[f(x)]^2\,dx$; this is usually written $\pi y^2\,dx$, since $y = f(x)$. This value of the element may readily be deduced from the figure on supposing that Q_1Q_2 is an infinitesimal distance.

If an arc of $y = f(x)$ between the points for which $y = c$ and $y = d$ revolves about the y-axis, it can be shown in a similar way that the element of volume is $\pi x^2\,dy$, and that the total volume generated by the revolution is

$$\pi \int_c^d x^2\,dy.$$

Before integrating it will be necessary to express x^2 in terms of y.

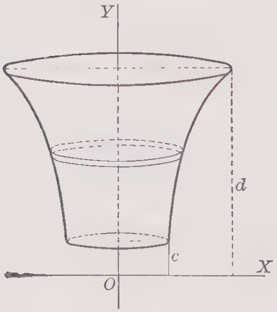

Fig. 25.

Ex. 1. Find the volume of the right cone generated by revolving about the x-axis the line joining the origin and the point (h, a).

Let M be the point (h, a). The equation of OM is

$$ax = hy.$$

The element of volume is $\pi y^2\,dx$. Hence,

$$\text{volume } OMN = \pi\int_0^h y^2\,dx$$

$$= \pi\int_0^h \frac{a^2x^2}{h^2}\,dx.$$

$$= \frac{\pi a^2 h}{3}.$$

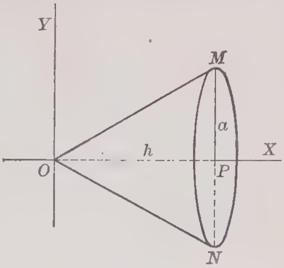

FIG. 26.

This may be interpreted: the volume of the right circular cone OMN is equal to one third the area of the base by its altitude.

Ex. 2. Find the volume of the cone generated by the same line on revolving about the y-axis.

In this case, the element of volume is $\pi x^2\,dy$. Hence,

$$\text{volume } OMN = \pi\int_0^a x^2\,dy$$

$$= \pi\int_0^a \frac{h^2y^2}{a^2}\,dy$$

$$= \frac{\pi\,ah^2}{3}.$$

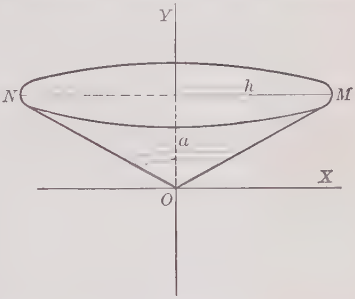

FIG. 27.

Ex. 3. Find the volume of the solid generated by revolving the arc of the parabola $y^2 = 4px$ between the origin and the point for which $x = x_1$, about the x-axis.

In this case,

$$\text{volume } OPP_1 = \pi\int_0^{x_1} y^2\,dx$$

$$= \pi\int_0^{x_1} 4px\,dx$$

$$= 2\pi p x_1^2;$$

or, since $y_1^2 = 4px_1$,

$$OPP_1 = \frac{\pi y_1^2 x_1}{2}.$$

Hence, the volume is one half the volume of the circumscribing cylinder.

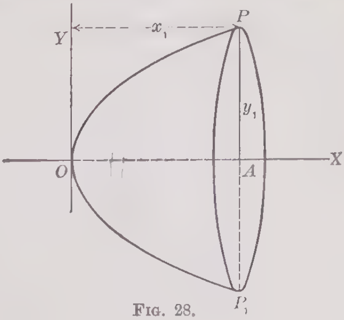

FIG. 28.

Ex. **4**. Find the volume generated by revolving the arc in Ex. 3 about the y-axis.

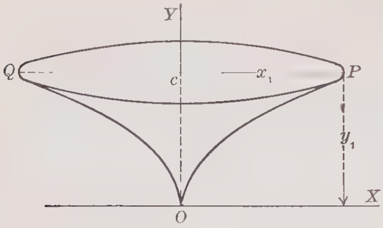

FIG. 29.

In this case,

$$\text{volume } OPQ = \pi \int_0^{y_1} x^2\,dy$$

$$= \pi \int_0^{y_1} \frac{y^4}{16\,p^2}\,dy$$

$$= \frac{\pi y_1^5}{5 \cdot 16\,p^2};$$

or, since $y_1^2 = 4\,px_1$,

$$\text{volume } OPQ = \tfrac{1}{5}\,\pi y_1 x_1^2.$$

Hence, the volume required is one fifth the volume of the cylinder of base PQ and height CO.

Ex. **5**. Find the volume of the solid generated by the revolution about the x-axis of the arc of the curve $y = (x + 1)(x + 2)$ between the points whose abscissas are 1, 2.

Ex. **6**. Find the volume of the cone generated by the revolution about the x-axis of the parts of each of the following lines intercepted between the axes:

(*a*) $2\,x + y = 10$; (*c*) $4\,x - 5\,y + 3 = 0$;
(*b*) $7\,x + 2\,y + 3 = 0$; (*d*) $3\,x - 8\,y = 5$.

Ex. **7**. Find the volume of the cone generated by the revolution about the y-axis of the parts of each of the following lines intercepted between the axes:

(*a*) $4\,x + 3\,y = 6$; (*c*) $5\,x - 7\,y + 35 = 0$;
(*b*) $3\,x - 4\,y = 6$; (*d*) $2\,x + 6\,y + 9 = 0$.

Ex. **8**. Find the volume of revolution about the x-axis of the arcs of the following curves between the assigned limits:

(*a*) $y^2 = x^3,\ x = 0,\ x = 2$; (*b*) $(a^2 + x^2)\,y^4 = a^4,\ x = 0,\ x = a$.

Ex. **9**. Find the volume of the solid generated by the revolution about the x-axis of the curve $y^3 = cx$ from the origin to the point whose abscissa is x_1.

Ex. **10**. Find the volume of the solid generated by the revolution of the same arc as in Ex. 9 about the y-axis.

Ex. **11**. Find the volume of the solid generated by the revolution about the y-axis of an arc of the curve in Ex. 9 from the origin to $y = y_1$.

Ex. **12**. Find the volume of the prolate spheroid generated by the revolution of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the x-axis.

31. On the graphical representation of a definite integral. In Arts. 4, 6, attention has been drawn to the principle that any definite integral, whether it denotes volume, length, surface, force, mass, work, etc., may be graphically represented by an area.* A simple illustration may put this in a clearer light.

Ex. Find the volume of the right cone generated by revolving about the x-axis the line drawn from the origin to the point (4, 1).

Let P be the point (4, 1), and let POQ be the cone of revolution. The equation of OP is $4y = x$.

Hence,

$$\text{vol. } POQ = \int_0^4 \pi y^2\,dx$$

$$= \int_0^4 \frac{\pi x^2}{16}\,dx \quad (1)$$

$$= \tfrac{4}{3}\pi.$$

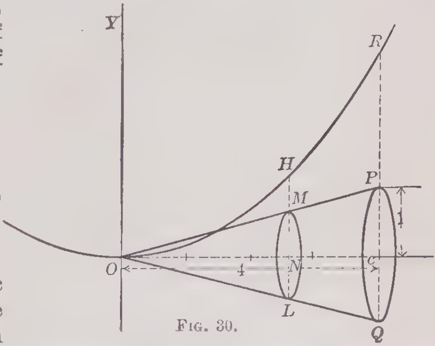

FIG. 30.

The volume is thus $\frac{4}{3}\pi$ cubic units of the same kind as the linear unit employed. In order to represent this volume graphically, draw the curve OHR whose equation is

$$y = \frac{\pi}{16}x^2,$$

$\frac{\pi}{16}x^2$ being the function of x under the sign of integration in (1); and draw the ordinate CR at $x = 4$. The area ROC graphically represents the volume POQ. For,

$$\text{area } ROC = \int_0^4 y\,dx$$

$$= \int_0^4 \frac{\pi x^2}{16}\,dx \qquad (2)$$

$$= \tfrac{4}{3}\pi.$$

Equations (1) and (2) show that the *number* of cubic units which indicates the volume of POQ is the same as the *number* of square units which indicates the area of ROC. In the same way, if the ordinate NH be drawn at any point N, for which $x = a$, say, it can be shown that $\frac{1}{48}\pi a^3$ denotes both the number of cubic units in the cone MOL and the number of square units in the area HON. It is thus apparent that the number of cubic units in $LMPQ$ is the same as the number of square units in $NHRC$, namely $\frac{1}{48}\pi(64 - a^3)$. Hence,

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* On account of this property the process of integration was called by Newton and the earlier writers "the method of quadratures."

$$\begin{aligned}&\text{vol. } POQ : \text{vol. } MOL : \text{vol. } LMPQ \\ &= \text{area } ROC : \text{area } HON : \text{area } NHRC. \qquad (3)\end{aligned}$$

If the curve $y = \frac{1}{16}m\pi x^2$ be drawn, the numbers which indicate the areas will be m times the numbers which indicate the volumes of the corresponding sections of the cone. But the ratio of any two right sections of the cone will be the same as the ratio of the two corresponding areas, and proportion (3) will still hold. The curve $y = \frac{1}{16}m\pi x^2$ can therefore be used to represent the volume. It is sometimes well to use a multiplier m for the sake of convenience in plotting the curve that will graphically represent the integral.

Note. If the first integral curve (see Art. 15) of OHR, namely,

$$y = \int_0^x \frac{\pi}{16}x^2 = \frac{1}{48}\pi x^3,$$

be drawn, its ordinates represent both the areas of the segments of OHR and the volumes of the segments of the cone POQ measured from O.

32. Derivation of the equations of certain curves. Oftentimes, when a curve is described by some property belonging to it, the formal analytic statement of the property involves differential coefficients. In these cases the derivation of the equation of the curve consists in finding a relation between the coördinates which will be free from differentials. Examples of this have been given in Art. 12. A few additional simple instances are introduced here. In the larger number of cases the derivation of the equation of the curve will require a greater knowledge of differential equations than the student possesses at this stage; and hence further problems of this kind will be deferred until Chap. XIII.

Ex. 1. Determine the curve whose subtangent is n times the abscissa of the point of contact. Find the particular curve which passes through the point (5, 4).

Let (x, y) be any point on the curve. The subtangent is $y\frac{dx}{dy}$. By the given condition,

$$y\frac{dx}{dy} = nx.$$

This may be written, $$\frac{dx}{x} = \frac{n\,dy}{y}.$$

Integrating, $\log c + \log x = n \log y$;

whence, $$y^n = cx. \qquad (1)$$

All the curves obtained by varying c, satisfy the given condition.

If one of the curves passes through the point (5, 4), for instance,

$$4^n = 5\,c. \qquad (2)$$

Substitution in (1) of the value of c from (2) gives

$$5\,y^n = 4^n x,$$

as the equation of the particular curve through (5, 4).

What curves have the given property for $n = 1$? $n = 2$? $n = \frac{3}{2}$? $n = \frac{1}{2}$? $n = \frac{2}{3}$?

Ex. **2**. Find the curves in which the polar subnormal is proportional to (is k times) the sine of the vectorial angle. What particular curves pass through the point $(0, 2\,\pi)$?

The polar subnormal is $\frac{dr}{d\theta}$. By the given condition,

$$\frac{dr}{d\theta} = k \sin \theta.$$

Integrating, $$r = c - k \cos \theta.$$

For the curve that passes through $(0, 2\,\pi)$, $0 = c - k$; whence $c = k$. Hence, the equation of the particular curve required is

$$r = k(1 - \cos \theta),$$

the equation of the cardioid.

Ex. **3**. Determine the curve in which the subtangent is n times the subnormal; and find the particular curve that passes through the point (2, 3).

Ex. **4**. Determine the curve in which the length of the subnormal is proportional to the square of the ordinate.

Ex. **5**. Determine the curve in which the subnormal is proportional to (is k times) the nth power of the abscissa.

Ex. **6**. Find the curve in which, for any point, the length of the polar subtangent is proportional to (is k times) the length of the radius vector.

Ex. **7**. Find the curve in which the angle between the radius vector and the tangent at any point is n times the vectorial angle. What is the curve when $n = 1$? when $n = \frac{1}{2}$?

EXAMPLES ON CHAPTER IV

1. Find the area of the figure bounded by the curve $x^4 + ax^3 + a^2x^2 + b^3y = 0$, the x-axis, and the ordinates at $x = 0$, $x = a$.

2. Find the area inclosed by the curve $xy^2 = y^{\frac{5}{2}} + 2y^{\frac{3}{2}}$ and the lines $x = 0$, $y = 0$, $y = 1$.

3. Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

4. Find the area included between the catenary $y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$, the axes of coördinates, and the line $x = c$.

5. In the logarithmic curve $y = e^{ax}$ prove that the area between the curve, the axis of x, and any two ordinates is proportional to the difference between the ordinates.

6. Find the area included between the curve $y = \frac{x}{1 + x^2}$, and the line $y = \frac{x}{4}$.

7. Find the area bounded by the curve $y = x^3 + ax^2$, the x-axis, and

(*a*) the ordinates at $x = -a$, and $x = 0$;
(*b*) the ordinates at $x = 0$, $x = a$.

8. Find the area inclosed by the axis of x, and the curve $y = x - x^3$.

9. Find the entire area of the curve $y^2 = a^2x^2 - x^4$.

10. Find the area included between the curve $y^2(a^2 - x^2) = a^2x^2$ and its asymptote $x = a$.

11. Find the entire area contained between the curve $y^2(a^2 - x^2) = a^4$ and its asymptotes $x = a$, $x = -a$.

12. Find the area included by the curve $x^2y^2(x^2 - a^2) = a^6$ and its asymptote $x = a$.

13. Find the area of the loop of the curve $a^3y^2 = x^4(b + x)$.

14. Find the total area bounded by the curve $a^4y^2 + b^2x^4 = a^2b^2x^2$.

15. Find the volume of the solid generated by the revolution about the x-axis of:

(*a*) $y^2 = x^3 - x^2$ between the ordinates $x = 1$, $x = 2$;
(*b*) $(a^2 - x^2)y^4 = a^6$ between the curve and its asymptotes $x = a$, $x = -a$.

16. Find the volume generated by the revolution about either axis of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

17. Find the volume of the solid generated by the revolution about either axis of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

18. Find the volume of the solid generated by the revolution about the y-axis of that portion of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ between the lines $x = a$, $x = -a$.

19. Find the volume generated by the revolution of the cissoid $y^2 = \frac{x^3}{2a - x}$ about the x-axis from the origin to $x = a$.

20. Find the volume generated by the revolution of the cissoid in Ex. 19 about its asymptote $x = 2a$. [Reference may be made to the table of integrals in the Appendix.]

21. Find the volume of the frustum of a cone obtained by rotating about the x-axis the line joining the points $(-4, 1)$ and $(3, 6)$.

22. The hyperbola $xy = c^2$ revolves about the axis of y. Show that the volume generated by the infinite branch extended from the vertex (c, c) towards the y-axis is equal to the volume of the cylinder generated by the revolution of the ordinate at the vertex about the y-axis. Show that the area which generates the first volume is infinite.

23. Find the volume of the ring generated by the circle $x^2 + y^2 = 25$ revolving about the line $x = 7$.

24. Show that in the solid generated by the revolution of the rectangular hyperbola $x^2 - y^2 = a^2$ about the x-axis, the volume of a segment of height a measured from the vertex, is equal to that of a sphere of radius a.

25. Show that the volume generated by the revolution of one semi-undulation of the curve $y = b \sin \frac{x}{a}$ about the x-axis is one half that of the circumscribing cylinder.

26. The figure bounded by a quadrant of a circle of radius a, and the tangents at its extremities, revolves about one of these tangents; find the volume of the solid thus generated.

CHAPTER V

RATIONAL FRACTIONS

33. A rational fraction is one in which the numerator and denominator are rational integral functions of the variables. The fraction is *proper* when the degree of the numerator is lower than that of the denominator. If the degree of the numerator is greater than that of the denominator, division can be carried on until the remainder is of less degree than the denominator. Suppose that N, D are rational integral functions of x, and that the degree of N is greater than that of D. By division,

$$\frac{N}{D} = Q + \frac{R}{D},$$

in which R is of lower degree than D; and, therefore,

$$\int \frac{N}{D} dx = \int Q\, dx + \int \frac{R}{D} dx.$$

In order to integrate the proper fraction $\frac{R}{D}$ it is often necessary to resolve it into partial fractions. It can be shown that any proper rational fraction can be decomposed into partial fractions of the types

$$\frac{A}{x-a}, \quad \frac{B}{(x-a)^r}, \quad \frac{Cx+G}{x^2+px+q}, \quad \frac{Ex+F}{(x^2+px+q)^s},$$

in which A, B, C, G, E, F are constants, r, s positive integers, and x^2+px+q is an expression whose factors are imaginary. For the proof of this and the related theory, reference may be made to works on algebra.* Here nothing more is done than to work some examples in the principal cases that occur in practice.†

34. Case I. *When the denominator can be resolved into factors of the first degree, all of which are real and different.*

Ex. 1. Find $\int \frac{x^4 - 7x^2 + 6x - 6}{x^3 - x^2 - 6x} dx.$

On division, $\frac{x^4 - 7x^2 + 6x - 6}{x^3 - x^2 - 6x} = x + 1 + \frac{6(2x-1)}{x^3 - x^2 - 6x};$

or, $x + 1 + \frac{6(2x-1)}{x(x-3)(x+2)}.$

Put
$$\frac{6(2x-1)}{x(x-3)(x+2)} \equiv \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}, \tag{1}$$
in which A, B, C are constants to be determined.

Clearing of fractions,

$$6(2x-1) \equiv A(x-3)(x+2) + Bx(x+2) + Cx(x-3).$$

Since this is an identical equation, the coefficients of the same powers of x in each member are equal. On equating the coefficients of like powers of x, it is found that

$$A + B + C = 0,$$
$$-A + 2B - 3C = 12,$$
$$-6A = -6.$$

On solving these equations for A, B, C, there results

$$A = 1,\ B = 2,\ C = -3.$$

Therefore, after substituting these values in (1),

$$\int \frac{x^4 - 7x^2 + 6x - 6}{x^3 - x^2 - 6x} dx = \int \left(x + 1 + \frac{1}{x} + \frac{2}{x-3} - \frac{3}{x+2}\right) dx$$
$$= \tfrac{1}{2}x^2 + x + \log x + 2\log(x-3) - 3\log(x+2)$$
$$= \tfrac{1}{2}x(x+2) + \log \frac{x(x-3)^2}{(x+2)^3}.$$

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* See Chrystal's Algebra, Part I., Chap. VIII., Arts. 6–8.

† A few remarks on the decomposition of rational fractions are given in Note A, Appendix.

A shorter method for calculating A, B, C, could have been employed in the example just solved.

Since
$$\frac{6\,(2\,x-1)}{x\,(x-3)\,(x+2)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+2}$$
is an identity, it is true for any value of x.

Clearing of fractions,
$$6\,(2\,x-1)=A\,(x-3)(x+2)+Bx\,(x+2)+Cx\,(x-3).$$

On letting the factor $x=0$, $A=1$;

on letting the factor $x-3=0$, or $x=3$, $B=2$;

and on letting the factor $x+2=0$, or $x=-2$, $C=-3$.

Ex. 2. $\int\frac{5\,dx}{x^2-x-6}$.

Ex. 3. $\int\frac{(5\,x+1)dx}{x^2-2\,x-35}$.

Ex. 4. $\int\frac{(4-3\,x)dx}{x^2-3\,x+2}$.

Ex. 5. $\int\frac{dx}{x^3-x}$.

Ex. 6. $\int\frac{(x^2+1)dx}{x(x^2-1)}$.

Ex. 7. $\int\frac{(a-b)x\,dx}{x^2-(a+b)x+ab}$.

Ex. 8. $\int\frac{(3\,x+1)dx}{2\,x^2+3\,x-2}$.

Ex. 9. $\int\frac{(1-3\,x^2)dx}{3\,x-x^3}$.

Ex. 10. $\int\frac{(3\,x-1)dx}{x^2+x-6}$.

Ex. 11. $\int\frac{(1+x^2)dx}{x-x^3}$.

Ex. 12. $\int\frac{2\sqrt{3}\,dx}{x^2-4\,x+1}$.

Ex. 13. $\int\frac{2\,x^3-6\,x^2-4\,x-11}{x^2-3\,x-4}dx$.

Ex. 14. $\int\frac{x^3+7\,x^2+6\,x-6}{x^2+2\,x}dx$.

Ex. 15. $\int\frac{x^2+pq}{x(x-p)(x+q)}dx$.

Ex. 16. $\int\frac{x^2-3\,x+3}{(x-1)(x-2)}dx$.

Ex. 17. $\int\frac{(x+1)dx}{x^2+2\,x-4}$.

Ex. 18. $\int\frac{(8\,x^3-31\,x^2+41\,x-6)dx}{12(x^4-6\,x^3+11\,x^2-6\,x)}$.

Ex. 19. $\int\frac{(2\,x^2-5)dx}{x^4-5\,x^2+6}$.

Ex. 20. $\int\frac{dx}{x^7-7\,x^5+14\,x^3-8\,x}$.

Ex. 21. $\int\frac{(x-a)(x-b)(x-c)}{(x-\alpha)(x-\beta)(x-\gamma)}dx$.

[Suggestion. — Assume the fraction equal to $1+\frac{A}{x-\alpha}+$ etc.]

35. CASE II. *When the denominator can be resolved into linear factors, all of which are real and some of which are repeated.*

Ex. 1. $\int \frac{6x^3 - 8x^2 - 4x + 1}{x^4 - 2x^3 + x^2}\,dx.$

Let $$\frac{6x^3 - 8x^2 - 4x + 1}{x^2(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2},$$

in which A, B, C, D, are constants to be determined.

Clearing of fractions,

$$6x^3 - 8x^2 - 4x + 1 \equiv Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2.$$

Equating coefficients of like powers,

$$\begin{aligned} A + C &= 6, \\ -2A + B - C + D &= -8, \\ A - 2B &= -4, \\ B &= 1. \end{aligned}$$

On solving these equations it is found that $A = -2$, $B = 1$, $C = 8$, $D = -5$. Therefore,

$$\begin{aligned} \int \frac{6x^3 - 8x^2 - 4x + 1}{x^2(x-1)^2}\,dx &= \int \left(-\frac{2}{x} + \frac{1}{x^2} + \frac{8}{x-1} - \frac{5}{(x-1)^2} \right) dx \\ &= -2\log x - \frac{1}{x} + 8\log(x-1) + \frac{5}{x-1} \\ &= \log\frac{(x-1)^8}{x^2} + \frac{4x+1}{x(x-1)}. \end{aligned}$$

Ex. 2. $\int \frac{x\,dx}{(x+1)^2}.$

Ex. 3. $\int \frac{(x-1)\,dx}{(x-3)^2}.$

Ex. 4. $\int \frac{2(x+2)\,dx}{(2x+1)^2}.$

Ex. 5. $\int \left(\frac{a}{x+a} - \frac{bx}{(x+b)^2} \right) dx.$

Ex. 6. $\int \frac{2\,dx}{(3\sqrt{5} - 2 - x)^3}.$

Ex. 7. $\int \frac{3x(x-4)\,dx}{(x-3)^3}.$

Ex. 8. $\int \frac{(x-1)\,dx}{x^3 + 2x^2 + x}.$

Ex. 9. $\int \frac{ax^2\,dx}{(x+a)^3}.$

Ex. 10. $\int \frac{4x^2 + x - 2}{x^3 - x^2}\,dx.$

Ex. 11. $\int \frac{(2x-5)\,dx}{(x+3)(x+1)^2}.$

Ex. 12. $\int \frac{x^2\,dx}{x^3+5x^2+8x+4}$.

Ex. 13. $\int \frac{dx}{(x^2-2)^2}$.

Ex. 14. $\int \frac{9(2+4x-x^2)\,dx}{(x^2-x-2)^3}$.

Ex. 15. $\int \frac{(x^3+1)\,dx}{x(x-1)^3}$.

36. CASE III. *When the denominator contains quadratic factors, the linear factors of which are imaginary.* This case can be subdivided into the two following:

(*a*) When all such quadratic factors are different.

(*b*) When one or more of them is repeated.

The latter case seldom occurs in practice. For each power, from the first to the nth, of these quadratic factors, a numerator of the form $Mx + N$, in which M, N are constants, should be assumed.

Ex. 1. Find $\int \frac{4\,dx}{x^3+4x}$.

Assume $$\frac{4}{x(x^2+4)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

Clearing of fractions,

$$4 \equiv A(x^2+4) + x(Bx+C).$$

Equating coefficients of like terms,

$$A + B = 0,$$
$$C = 0,$$
$$4A = 4.$$

The solution of these equations gives

$$A = 1, \quad B = -1, \quad C = 0.$$

Hence, $$\int \frac{4\,dx}{x^3+4x} = \int \left(\frac{1}{x} - \frac{x}{x^2+4}\right)dx$$
$$= \log x - \tfrac{1}{2}\log(x^2+4)$$
$$= \log \frac{x}{\sqrt{x^2+4}}.$$

Ex. 2. Find $\int \frac{x^3+1}{(x-2)(x^2-2x+3)^2}\,dx$.

Assume $$\frac{x^3+1}{(x-2)(x^2-2x+3)^2} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3} + \frac{Dx+E}{(x^2-2x+3)^2}.$$

Clearing of fractions,

$$x^3+1 \equiv A(x^2+2x+3)^2+(Bx+C)(x-2)(x^2-2x+3)+(Dx+E)(x-2).$$

Equating coefficients of like powers of x,

$$\begin{aligned} A+B&=0,\\ -4A+C-4B&=1,\\ 10A+7B-4C+D&=0,\\ -12A-6B+7C+E-2D&=0,\\ 9A-6C-2E&=1. \end{aligned}$$

The solution of these equations is, $A=1,\ B=-1,\ C=1,\ D=1,\ E=1.$

Hence,

$$\int\frac{(x^3+1)dx}{(x-2)(x^2-2x+3)^2}=\int\left(\frac{1}{x-2}-\frac{x-1}{x^2-2x+3}+\frac{x+1}{(x^2-2x+3)^2}\right)dx$$

$$=\log\frac{x-2}{\sqrt{x^2-2x+3}}-\frac{1}{2(x^2-2x+3)}+\int\frac{2\,dx}{(x^2-2x+3)^2}.$$

The last integral can be found by means of reduction formulæ to be given in the next chapter.

Ex. 3. $\int\frac{(x+1)^2dx}{x^3+x}.$

Ex. 4. $\int\frac{2x^2+x+3}{(x+1)(x^2+1)}dx.$

Ex. 5. $\int\frac{3x^5+6x^3+x^2+2x+2}{3x^3+6}dx.$

Ex. 6. $\int\frac{(x^2+2)dx}{(x^2+1)^2}.$

Ex. 7. $\int\frac{(2x^2+x+2)dx}{(3x+2)(2x^2+4x+4)}.$

Ex. 8. $\int\frac{(3x^2-17x+33)dx}{x^3-6x^2+11x}.$

Ex. 9. $\int\frac{x^5-3x^4-22x^3+17x^2-23x+20}{(x+4)(x^2+1)}dx.$

Ex. 10. $\int\frac{x^4+7x^2+13}{(x^2+3)^3}dx.$

Ex. 11. $\int\frac{3x^4+3x^3+30x^2+17x+75}{3(x^2+5)^3}dx.$

Ex. 12. $\int\frac{x^3-1}{x^3+3x}dx.$

Ex. 13. $\int\frac{(x^3-6)dx}{x^4+6x^2+8}.$

Ex. 14. $\int\frac{3x-7}{x^3+x^2+4x+4}dx=\log\frac{x^2+4}{(x+1)^2}+\frac{1}{2}\tan^{-1}\frac{x}{2}.$

CHAPTER VI

IRRATIONAL FUNCTIONS

37. The integrals of irrational functions can be found in only a few cases. In some instances, by means of substitutions, these functions can be changed into equivalent functions that either are in the list of fundamental integrals, or are rational, and therefore integrable. In other instances the integral can be found by means of a formula of reduction. A few irrational forms have been discussed in preceding articles.

38. The reciprocal substitution. Sometimes the integration of an irrational function is facilitated by the substitution

$$x = \frac{1}{t}, \; dx = -\frac{1}{t^2}dt.$$

Ex. 1. Find $\int \frac{\sqrt{a^2 - x^2}}{x^4}dx.$

On putting $x = \frac{1}{t}$, $\int \frac{\sqrt{a^2 - x^2}}{x^4}dx = -\int (a^2t^2 - 1)^{\frac{1}{2}}t\,dt$

$$= -\frac{(a^2t^2 - 1)^{\frac{3}{2}}}{3\,a^2}$$

$$= -\frac{(a^2 - x^2)^{\frac{3}{2}}}{3\,a^2x^3}.$$

Ex. 2. $\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}}.$

Ex. 3. $\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}.$

Ex. 4. $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}.$

Ex. 5. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}}.$

Ex. 6. $\int \frac{dx}{x^2\sqrt{x^2 + a^2}}.$

Ex. 7. $\int \frac{\sqrt{x^2 - a^2}}{x^4}dx.$

Ex. 8. $\int \frac{dx}{x\sqrt{a^2+x^2}}$.

Ex. 9. $\int \frac{dx}{x\sqrt{a^2-x^2}}$.

Ex. 10. $\int \frac{dx}{x\sqrt{x^2-a^2}}$.

Ex. 11. $\int \frac{dx}{x\sqrt{2\,ax-x^2}}$.

Ex. 12. $\int \frac{x\,dx}{(2\,ax-x^2)^{\frac{3}{2}}}$.

Ex. 13. $\int \frac{\sqrt{2\,ax-x^2}}{x^3}\,dx$.

39. Trigonometric substitutions. Although the integration of trigonometric functions is not discussed until the next chapter, it may be stated here that trigonometric substitutions sometimes aid the integration of irrational functions. The following substitutions may be tried:

(*a*) $x = a\sin\theta$ for functions that involve $\sqrt{a^2-x^2}$,

(*b*) $x = a\tan\theta$ for functions that involve $\sqrt{a^2+x^2}$,

(*c*) $x = a\sec\theta$ for functions that involve $\sqrt{x^2-a^2}$.

Ex. Find $\int\sqrt{a^2-x^2}\,dx$. (See Ex. 1, Art. 23.)

On assuming $x = a\sin\theta,\ dx = a\cos\theta\,d\theta,$

and
$$\int\sqrt{a^2-x^2}\,dx = a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}\int(1+\cos 2\,\theta)d\theta$$
$$=\frac{a^2}{2}(\theta+\tfrac{1}{2}\sin 2\,\theta) = \frac{a^2}{2}(\theta+\sin\theta\cos\theta)$$
$$=\frac{1}{2}\left(a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}\right).$$

40. Expressions containing fractional powers of $a+bx$ only. If n is the least common multiple of the denominators of the powers of $a+bx$, these expressions can be reduced to the form

$$F(x,\ \sqrt[n]{a+bx}).$$

If $F(u,\ v)$ is a rational function of u, v, then $\int F(x,\ \sqrt[n]{a+bx})\,dx$ can be changed into a rational form by means of the substitution

$$a+bx = z^n.$$

For then $$x = \frac{z^n - a}{b}, \quad dx = \frac{nz^{n-1}}{b}dz,$$

and hence, $$\int F(x, \sqrt[n]{a + bx})\,dx = \frac{n}{b}\int F\left(\frac{z^n - a}{b}, z\right)z^{n-1}\,dz.$$

Expressions that contain fractional powers of x only, belong to this class. This is apparent on putting $a = 0$ in $a + bx$. If n is the least common multiple of the denominators of the exponents of x, the function can be changed into a rational form by the substitution

$$x = z^n.$$

Ex. 1. Find $\int \frac{dx}{x\sqrt{a^2 + bx}}$.

On making the substitution $a^2 + bx = z^2$, $x = \frac{z^2 - a^2}{b}$, $dx = \frac{2z}{b}dz$.

Hence, $$\int \frac{dx}{x\sqrt{a^2 + bx}} = 2\int \frac{dz}{z^2 - a^2}$$
$$= \frac{1}{a}\log\frac{z - a}{z + a}$$
$$= \frac{1}{a}\log\frac{\sqrt{a^2 + bx} - a}{\sqrt{a^2 + bx} + a}.$$

Ex. 2. Find $\int \frac{x^{\frac{1}{2}}}{1 + x^{\frac{1}{3}}}dx$.

The L. C. M. of the denominators of the exponents is 6. If $x = z^6$, then $dx = 6z^5\,dz$, and

$$\int \frac{x^{\frac{1}{2}}}{1 + x^{\frac{1}{3}}}dx = 6\int \frac{z^8}{1 + z^2}dz$$
$$= 6\int \left(z^6 - z^4 + z^2 - 1 + \frac{1}{1 + z^2}\right)dz$$
$$= 6\left(\frac{z^7}{7} - \frac{z^5}{5} + \frac{z^3}{3} - z + \tan^{-1} z\right)$$
$$= 6x^{\frac{1}{6}}(\tfrac{1}{7}x - \tfrac{1}{5}x^{\frac{2}{3}} + \tfrac{1}{3}x^{\frac{1}{3}} - 1) + 6\tan^{-1} x^{\frac{1}{6}}.$$

Ex. 3. Find $\int \frac{\sqrt[3]{x}}{x - 1}\,dx$.

Ex. 4. Find $\int \frac{3 + 5(x^{\frac{2}{15}} + x^{\frac{1}{3}})dx}{15(x + x^{\frac{4}{5}})}$.

Ex. 5. $\displaystyle\int \frac{x^{\frac{1}{7}} + \sqrt{x}}{x^{\frac{8}{7}} + x^{\frac{15}{14}}}\,dx.$

Ex. 6. $\displaystyle\int \frac{(x^{\frac{1}{4}} - 2\,x^{\frac{1}{6}})\,dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}.$

Ex. 7. $\displaystyle\frac{\sqrt[4]{x^3} - 7\sqrt[3]{x^2} + 12\sqrt{x}}{x(\sqrt[3]{x} - \sqrt[6]{x})}\,dx.$

Ex. 8. $\displaystyle\int x\sqrt{a + bx}\,dx.$

Ex. 9. $\displaystyle\int \frac{dx}{\sqrt{x+1}\,(3 + 2\sqrt{x+1})}.$

Ex. 10. $\displaystyle\int \frac{\sqrt{x+1} + 1}{\sqrt{x+1} - 1}\,dx.$

Ex. 11. $\displaystyle\int (3 - x)\sqrt[3]{(c - x)^2}\,dx.$

Ex. 12. $\displaystyle\int \frac{x\,dx}{(2\,x + 3)^{\frac{4}{3}}}.$

41. Functions of the form $f\{x^2, (a + bx^2)^{\frac{m}{n}}\}\cdot x\,dx$, in which m, n are integers. If $f(u, v)$ is a rational function of u, v, these functions can be rationalized by means of the substitution

$$a + bx^2 = z^n.$$

For then, $2\,bx\,dx = nz^{n-1}dz$, $x^2 = \dfrac{z^n - a}{b}$, and the function becomes

$$\frac{n}{2\,b} f\left(\frac{z^n - a}{b},\ z^n\right) z^{n-1}\,dz.$$

Ex. 1. Find $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}}.$ (See Ex. 10, Art. 38.)

This belongs to the form above, since $\dfrac{1}{x\sqrt{x^2 - a^2}} = \dfrac{x}{x^2\sqrt{x^2 - a^2}}.$

On putting $x^2 - a^2 = z^2$, $x\,dx = z\,dz$, $x^2 = z^2 + a^2$, and

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \int \frac{dz}{z^2 + a^2}$$

$$= \frac{1}{a}\tan^{-1}\frac{z}{a}$$

$$= \frac{1}{a}\tan^{-1}\frac{\sqrt{x^2 - a^2}}{a} = \frac{1}{a}\cos^{-1}\frac{a}{x}.$$

Ex. 2. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + a^2}}.$ (See Ex. 8, Art. 38.)

Ex. 3. $\displaystyle\int \frac{x^3\,dx}{\sqrt{1 - x^2}}.$

Ex. 4. $\int \frac{x\,dx}{(1+x^2)\sqrt{1-x^2}}$. Ex. 5. $\int \frac{x^3\,dx}{(2+x^2)\sqrt{x^2+5}}$.

Ex. 6. If $f(u, v)$ is a rational function of u, v, show that

$$f\left(x,\ \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$$

can be rationalized by means of the substitution $\frac{ax+b}{cx+d} = z^n$.

42. Functions of the form $F(x,\ \sqrt{x^2+ax+b})dx$, $F(u,\ v)$ being a rational function of u, v. If the radical be $\sqrt{mx^2+px+q}$, it can be written $\sqrt{m}\sqrt{x^2+\frac{p}{m}x+\frac{q}{m}}$.

The given function can be rationalized by assuming that

$$\sqrt{x^2+ax+b} = z - x,$$

and then changing the variable from x to z.

For, squaring and solving for x,

$$x = \frac{z^2-b}{a+2z},$$

whence, $$z - x = \frac{z^2+az+b}{a+2z},$$

and $$dx = \frac{2(z^2+az+b)}{(a+2z)^2}dz.$$

Therefore, on substitution,

$$\int F(x,\ \sqrt{x^2+ax+b})\,dx = 2\int F\left(\frac{z^2-b}{a+2z},\ \frac{z^2+az+b}{a+2z}\right)\frac{z^2+az+b}{(a+2z)^2}\,dz.$$

Ex. $\int \frac{dx}{x\sqrt{x^2+x+1}}$.

Assume $$\sqrt{x^2+x+1} = z - x.$$

Squaring and solving for x, $$x = \frac{z^2-1}{1+2z}.$$

Hence,
$$dx = \frac{2(z^2+z+1)}{(1+2z)^2}\,dz,$$

and
$$\sqrt{x^2+x+1} = z - x = \frac{z^2+z+1}{1+2z}.$$

On substitution,
$$\int \frac{dx}{x\sqrt{x^2+x+1}} = 2\int \frac{dz}{z^2-1}$$
$$= \log\frac{z-1}{z+1}$$
$$= \log\frac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}.$$

43. Functions of the form $f(x,\ \sqrt{-x^2+ax+b})\,dx$, $f(u,\ v)$ being a rational function of u, v. If the radical be $\sqrt{-mx^2+px+q}$, it may be written $\sqrt{m}\sqrt{-x^2+\frac{p}{m}x+\frac{q}{m}}$. If the factors of $-x^2+ax+b$ are imaginary, $\sqrt{-x^2+ax+b}$ is imaginary. For, if one of the factors is $x-\alpha+i\beta$, the other must be $-(x-\alpha-i\beta)$, and hence,
$$-x^2+ax+b = -(x-\alpha+i\beta)(x-\alpha-i\beta)$$
$$= -\{(x-\alpha)^2+\beta^2\},$$
which is negative, and has an imaginary square root whatever x may be. Only cases in which the factors of $-x^2+ax+b$ are real will be considered here.

Let
$$-x^2+ax+b = (x-\alpha)(\beta-x).$$

Assume $\sqrt{-x^2+ax+b}$

or
$$\sqrt{(x-\alpha)(\beta-x)} = (x-\alpha)z.$$

Then, squaring,
$$\beta - x = (x-\alpha)z^2,$$
$$x = \frac{\alpha z^2+\beta}{1+z^2},$$

$$dx = \frac{2z(\alpha - \beta)}{(1+z^2)^2}\,dz,$$

and $\sqrt{-x^2+ax+b} = (x-\alpha)z = \dfrac{(\beta-\alpha)z}{1+z^2}$.

Hence, on making the substitutions,

$$F(x, \sqrt{-x^2+ax+b})\,dx = 2(\alpha-\beta)\,F\left(\frac{\alpha z^2+\beta}{1+z^2}, \frac{(\beta-\alpha)z}{1+z^2}\right)\frac{z\,dz}{(1+z^2)^2},$$

which is rational, and accordingly integrable.

Equally well, the substitution, $\sqrt{(x-\alpha)(\beta-x)} = (\beta-x)z$, might have been made.

It follows from this article and the preceding article that, if X is the indicated square root of an expression of the second degree in x, every rational function $f(x, X)$ is integrable.

Ex. 1. Find $\displaystyle\int\frac{dx}{x\sqrt{-x^2+5x-6}}$.

Assume

$$\sqrt{-x^2+5x-6} = \sqrt{(x-2)(3-x)} = (x-2)z.$$

From this, on squaring, $3 - x = (x-2)z^2$.

Hence,
$$x = \frac{2z^2+3}{z^2+1},$$

$$dx = -\frac{2z\,dz}{(z^2+1)^2},$$

and
$$\sqrt{-x^2+5x-6} = (x-2)z = \frac{z}{z^2+1}.$$

Therefore, on substitution,

$$\int\frac{dx}{x\sqrt{-x^2+5x-6}} = -2\int\frac{dz}{2z^2+3}$$
$$= -\sqrt{\tfrac{2}{3}}\tan^{-1}\sqrt{\tfrac{2}{3}}\,z$$
$$= -\sqrt{\frac{2}{3}}\tan^{-1}\sqrt{\frac{2(3-x)}{3(x-2)}}.$$

Ex. 2. $\displaystyle\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$. **Ex. 3.** $\displaystyle\int\frac{dx}{\sqrt{2x^2+3x+4}}$.

Ex. 4. $\int \frac{\sqrt{2x + x^2}}{x^2}\,dx.$

Ex. 5. $\int \frac{x+1}{\sqrt{x^2+x+1}}\,dx.$

Ex. 6. $\int \frac{dx}{x^3\sqrt{1-x^2}}.$

Ex. 7. $\int \frac{\sqrt{6x-x^2}}{x^2}\,dx.$

44. Particular functions involving $\sqrt{ax^2+bx+c}$.

(*a*) If a is positive,

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right).$$

(Ex. 43, Chap. III.)

(*b*) If a is negative, say $-a_1$,

$$\int \frac{dx}{\sqrt{-a_1x^2+bx+c}} = \frac{1}{\sqrt{a_1}}\sin^{-1}\frac{2a_1x-b}{\sqrt{b^2+4a_1c}}. \quad \text{(Ex. 44, Chap. III.)}$$

(*c*) $\int \frac{(Ax+B)}{\sqrt{ax^2+bx+c}}\,dx.$

Since $\frac{d}{dx}(ax^2+bx+c) = 2ax+b,$

and $$Ax+B = \frac{A}{2a}(2ax+b) + B - \frac{Ab}{2a},$$

$$\int \frac{(Ax+B)}{\sqrt{ax^2+bx+c}}\,dx = \frac{A}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \left(B - \frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= \frac{A}{a}\sqrt{ax^2+bx+c} + \text{an integral of the}$$

form (*a*) or (*b*) above.

Ex. 1. Find $\int \frac{(x+3)\,dx}{\sqrt{x^2+2x+3}}.$

Since $\frac{d}{dx}(x^2+2x+3) = 2x+2$, and $x+3 = \frac{1}{2}(2x+2)+2$,

$$\int \frac{(x+3)\,dx}{\sqrt{x^2+2x+3}} = \frac{1}{2}\int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+3}} + 2\int \frac{dx}{\sqrt{(x+1)^2+2}}$$

$$= \sqrt{x^2+2x+3} + 2\log\left(x+1+\sqrt{x^2+2x+3}\right).$$

Ex. 2. $\int \frac{5\,dx}{\sqrt{5x^2 - 2x + 7}}$. Ex. 6. $\int \frac{(5 + 3x)\,dx}{\sqrt{6 - 3x - 2x^2}}$

Ex. 3. $\int \frac{dx}{\sqrt{3x^2 - x + 1}}$. Ex. 7. $\int \sqrt{\frac{a + x}{a - x}}\,dx$.

Ex. 4. $\int \frac{dx}{\sqrt{-x^2 + 2x + 3}}$. Ex. 8. $\int \frac{(5x + 1)\,dx}{\sqrt{x^2 - 3}}$.

Ex. 5. $\int \frac{4\,dx}{\sqrt{-2x^2 + 3x + 4}}$. Ex. 9. $\int \frac{7x + 3}{\sqrt{3x^2 - 3x + 1}}\,dx$.

$$(d)\quad \int \frac{dx}{(x - \alpha)\sqrt{ax^2 + bx + c}}, \quad \int \frac{Mx + N}{(x - \alpha)\sqrt{ax^2 + bx + c}}\,dx,$$

M, N being constants.

On putting $x - \alpha = \frac{1}{z}$, $dx = -\frac{1}{z^2}\,dz$, and the first integral takes the form

$$-\int \frac{dz}{\sqrt{Az^2 + Bz + C}},$$

in which A, B, C, are constants. The first integral is thus reducible to (a) or (b).

Since $$Mx + N = M(x - \alpha) + N + M\alpha,$$

$$\int \frac{(Mx + N)}{(x - \alpha)\sqrt{ax^2 + bx + c}}\,dx = \int \frac{M(x - \alpha) + N + M\alpha}{(x - \alpha)\sqrt{ax^2 + bx + c}}\,dx$$

$$= M\int \frac{dx}{\sqrt{ax^2 + bx + c}} + (N + M\alpha)\int \frac{dx}{(x - \alpha)\sqrt{ax^2 + bx + c}}.$$

The two integrals in the second member have been considered above.

Ex. 10. Find $\int \frac{dx}{(x - 1)\sqrt{x^2 - 2x + 3}}$.

On putting $x - 1 = \frac{1}{z}$, $dx = -\frac{1}{z^2}\,dz$, and

$$\int \frac{dx}{(x - 1)\sqrt{x^2 - 2x + 3}} = -\int \frac{dz}{\sqrt{1 + 2z^2}} = -\frac{1}{\sqrt{2}} \log\left(z\sqrt{2} + \sqrt{1 + 2z^2}\right)$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{x^2 - 2x + 3} + \sqrt{2}}{x - 1}.$$

Ex. 11. Find $\int \frac{(3x+2)\,dx}{(x+2)\sqrt{2x^2+8x+10}}$.

Since $\frac{3x+2}{x+2} = 3 - \frac{4}{x+2}$, and $2x^2 + 8x + 10 = 2\{(x+2)^2 + 1\}$,

$$\int \frac{(3x+2)\,dx}{(x+2)\sqrt{2x^2+8x+10}} = \frac{3}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+2)^2+1}} - 2\sqrt{2} \int \frac{dx}{(x+2)\sqrt{(x+2)^2+1}}.$$

The integrals in the second member belong to forms already considered. Integration gives the result,

$$\frac{3}{\sqrt{2}} \log\left(x + 2 + \sqrt{x^2+4x+5}\right) + 2\sqrt{2} \log \frac{\sqrt{x^2+4x+5}+1}{x+2}.$$

Ex. 12. $\int \frac{(x^2+x+1)\,dx}{(x-3)\sqrt{5x^2-26x+34}}$. $\left(\text{Suggestion: } \frac{x^2+x+1}{x-3} = x + 4 + \frac{13}{x-3}.\right)$

Ex. 13. $\int \frac{dx}{x\sqrt{x^2+x+1}}$.

Ex. 14. $\int \frac{(2x+6)\,dx}{(x-1)\sqrt{1+2x-x^2}}$.

45. Integration of $x^m(a+bx^n)^p\,dx$: (*a*) by the method of undetermined coefficients; (*b*) by means of reduction formulæ. Some integrable functions are of the type $x^m(a+bx^n)^p$, in which a, b, m, n, p, are constants. The exponent n can always be positive. If the integration of an expression having this form is possible, it can always be effected by the method of "integration by parts." A shorter method, however, may sometimes be employed. The given integral is expressed in terms of a function of x not affected by a sign of integration, and of another integral which is easier to integrate than the original function.

(*a*) Rule. Put $\int x^m(a+bx^n)^p\,dx$ equal to a constant times one of the integrals

$$\int x^{m-n}(a+bx^n)^p\,dx, \quad \int x^{m+n}(a+bx^n)^p\,dx,$$

$$\int x^m(a+bx^n)^{p-1}\,dx, \quad \int x^m(a+bx^n)^{p+1}\,dx,$$

plus a constant times $x^{\lambda+1}(a+bx^n)^{\mu+1}$, in which λ, μ are the lowest indices of x and of $(a+bx^n)$ in the two expressions under the integration signs. Then determine the values of the constant coefficients thus introduced.

For example, on taking the first of the four integrals referred to in the rule,

$$\int x^m(a+bx^n)^p\,dx \equiv Ax^{m-n+1}(a+bx^n)^{p+1}+B\int x^{m-n}(a+bx^n)^p\,dx. \qquad (1)$$

Here λ, μ, the lowest indices of x, $(a+bx^n)$, under the sign $\int$, are $m-n$, p; and from this the term $Ax^{m-n+1}(a+bx^n)^{p+1}$ is derived. In order to determine the coefficients A, B, differentiate both members of equation (1). The result, after simplifying, is

$$x^n \equiv Ab\,(m+np+1)\,x^n + Aa\,(m-n+1)+B.$$

From this, on equating the coefficients of like powers of x,

$$A=\frac{1}{b\,(m+np+1)}, \qquad B=-\frac{a\,(m-n+1)}{b\,(m+np+1)}.$$

The substitution of these values in (1) gives

$$\int x^m\,(a+bx^n)^p\,dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b\,(m+np+1)} - \frac{a\,(m-n+1)}{b\,(m+np+1)}\int x^{m-n}\,(a+bx^n)^p\,dx. \qquad [\mathrm{A}]$$

Similarly, by connecting $\int x^m\,(a+bx^n)^p\,dx$ with the other integrals mentioned in the rule, the following results are obtained:

$$\int x^m\,(a+bx^n)^p\,dx = \frac{x^{m+1}(a+bx^n)^{p+1}}{a\,(m+1)} - \frac{b\,(m+n+np+1)}{a\,(m+1)}\int x^{m+n}(a+bx^n)^p\,dx. \qquad [\mathrm{B}]$$

$$\int x^m\,(a+bx^n)^p\,dx = \frac{x^{m+1}(a+bx^n)^p}{m+np+1} + \frac{anp}{m+np+1}\int x^m\,(a+bx^n)^{p-1}\,dx. \qquad [\mathrm{C}]$$

$$\int x^m(a+bx^n)^p\,dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)} + \frac{m+n+np+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}\,dx. \quad [\mathbf{D}]$$

In each of the four integrals with which $\int x^m(a+bx^n)^p\,dx$ may be connected by the rule, m is either increased or diminished by n, or else p is either increased or diminished by unity. The values of m and p will indicate the one of the four which is simpler than $\int x^m(a+bx^n)^p\,dx$, and may preferably be connected with it. Successive applications of the rule are necessary in some cases.

(b) The results **A**, **B**, **C**, **D**, can be used as formulæ of reduction. It is necessary only to make carefully the proper substitutions for m, n, p, in them. It is not necessary to memorize these formulæ, as they can be kept for reference.

It is well to be familiar with the process of deriving **A**, **B**, **C**, **D**, so that they can be readily obtained when required* if necessary. Formulæ **A**, **C** fail when $m+np+1=0$, **B** fails when $m+1=0$, and **D** fails when $p+1=0$. In these cases other methods can be used.

Ex. 1. Find $\int\frac{x^4\,dx}{\sqrt{a^2-x^2}}$.

Here, $m=4$, $n=2$, $p=-\frac{1}{2}$, $a=a^2$, $b=-1$.

On connecting this integral with $\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}$ by the rule in (a), it follows that

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* Another method of deriving these formulæ, that of integration by parts, is given in Note B, Appendix. Other formulæ of reduction can be obtained by connecting

$$\int x^m(a+bx^n)^p\,dx \text{ with } \int x^{m-n-1}(a+bx^n)^{p+1}\,dx \text{ and } \int x^{m+n-1}(a+bx^n)^{p-1}\,dx$$

in the manner described in the rule in (a). The student may derive the formulæ in this case as an exercise. (See Edwards' *Integral Calculus*, Art. 82.)

$$(1)\qquad \int \frac{x^4\,dx}{\sqrt{a^2-x^2}} \equiv Ax^3\sqrt{a^2-x^2} + B_1\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}.$$

The integral in the second member may be connected with $\int\frac{dx}{\sqrt{a^2-x^2}}$ thus,

$$(2)\qquad \int \frac{x^2\,dx}{\sqrt{a^2-x^2}} \equiv A_1x\sqrt{a^2-x^2} + C_1\int\frac{dx}{\sqrt{a^2-x^2}}.$$

Hence, on substitution in (1),

$$(3)\qquad \int \frac{x^4\,dx}{\sqrt{a^2-x^2}} \equiv Ax^3\sqrt{a^2-x^2} + Bx\sqrt{a^2-x^2} + C\int\frac{dx}{\sqrt{a^2-x^2}}.$$

On differentiating and simplifying,

$$x^4 = 3\,Ax^2(a^2-x^2) - Ax^4 + B(a^2-x^2) - Bx^2 + C.$$

On equating coefficients of like powers and solving for A, B, C, it is found that

$$A = -\tfrac{1}{4},\quad B = -\tfrac{3}{8}a^2,\quad C = \tfrac{3}{8}a^4.$$

Substitution of these values in (2) gives, since $\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$,

$$\int\frac{x^4\,dx}{\sqrt{a^2-x^2}} = \frac{1}{8}\left\{3\,a^4\sin^{-1}\frac{x}{a} - x(2\,x^2+3\,a^2)\sqrt{a^2-x^2}\right\}.$$

The coefficients A, B_1, might have been determined in (1), and A_1, C_1, might have been determined in (2).

The integral could also have been found by the application of formula [**A**] twice in succession.

Ex. 2. Find $\int\frac{dx}{(x^2+a^2)^k}$.

Here $m = 0$, $n = 2$, $p = -k$, $a = a^2$, $b = 1$.

Connecting the integral with $\int(x^2+a^2)^{-k+1}dx$,

$$(1)\qquad \int(x^2+a^2)^{-k}dx \equiv Ax(x^2+a^2)^{-k+1} + B\int(x^2+a^2)^{-k+1}dx.$$

On differentiating and dividing the resulting equation through by $(x^2-a^2)^{-k}$,

$$1 \equiv A(x^2+a^2) + 2\,Ax^2(1-k) + B(x^2+a^2).$$

On equating coefficients of like powers, and solving for A, B, it is found that

$$A = \frac{1}{2\,a^2(k-1)},\qquad B = \frac{2\,k-3}{2\,a^2(k-1)}.$$

Substitution of these values in (1) gives

$$\int\frac{dx}{(x^2+a^2)^k}=\frac{1}{2\,a^2(k-1)}\left\{\frac{x}{(x^2+a^2)^{k-1}}+(2\,k-3)\int\frac{dx}{(x^2+a^2)^{k-1}}.\right\}$$

This result might have been obtained by the application of formula [**D**].

Ex. 3. $\int\sqrt{a^2-x^2}\,dx.$ (See Ex. 1, Art. 23, Ex. Art. 39.)

Ex. 4. $\int\frac{x\,dx}{\sqrt{2\,ax-x^2}}.$ $[\sqrt{2\,ax-x^2}=x^{\frac{1}{2}}(2\,a-x)^{\frac{1}{2}}.]$

Ex. 5. $\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}.$ Ex. 6. $\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}.$ Ex. 7. $\int\frac{dx}{x^3\sqrt{a^2-x^2}}.$

Ex. 8. $\int x\sqrt{2\,ax-x^2}\,dx.$ [SUGGESTION. Put $\int x^{\frac{3}{2}}(2\,a-x)^{\frac{1}{2}}\,dx$

$= Ax^{\frac{5}{2}}(2\,a-x)^{\frac{1}{2}}+Bx^{\frac{3}{2}}(2\,a-x)^{\frac{1}{2}}+Cx^{\frac{1}{2}}(2\,a-x)^{\frac{1}{2}}+D\int x^{-\frac{1}{2}}(2\,a-x)^{-\frac{1}{2}}\,dx.]$

Ex. 9. $\int\frac{dx}{x^4\sqrt{a^2-x^2}}.$ Ex. 10. $\int\frac{x^3\,dx}{(x^2+a^2)^{\frac{1}{2}}}.$

Ex. 11. Show that

$$\int\frac{x^m\,dx}{\sqrt{a^2-x^2}}=-\frac{x^{m-1}\sqrt{a^2-x^2}}{m}+\frac{(m-1)a^2}{m}\int\frac{x^{m-2}\,dx}{\sqrt{a^2-x^2}}.$$

Ex. 12. Show that

$$\int x^m\sqrt{a^2-x^2}\,dx=\frac{x^{m+1}\sqrt{a^2-x^2}}{m+2}+\frac{a^2}{m+2}\int\frac{x^m\,dx}{\sqrt{a^2-x^2}}.$$

Ex. 13. Show that

$$\int\frac{dx}{x^n\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{(n-1)a^2x^{n-1}}+\frac{n-2}{(n-1)a^2}\int\frac{dx}{x^{n-2}\sqrt{a^2-x^2}}.$$

Ex. 14. Show that

$$\int\frac{x^m\,dx}{\sqrt{2\,ax-x^2}}=-\frac{x^{m-1}\sqrt{2\,ax-x^2}}{m}+\frac{(2\,m-1)a}{m}\int\frac{x^{m-1}\,dx}{\sqrt{2\,ax-x^2}}.$$

MISCELLANEOUS EXAMPLES.

1. $\dfrac{\sqrt{x}\,dx}{x\sqrt{x+a}}$.

2. $\displaystyle\int\frac{dx}{\sqrt{x-3}\sqrt{x-2}}$.

3. $\displaystyle\int\frac{M\,dx}{\sqrt{N^2x^2-Px+R}}$.

4. $\displaystyle\int\frac{\sqrt{x-a}+\sqrt{x-b}}{(x-a)\sqrt{x-b}}\,dx$.

5. $\displaystyle\int\frac{x^2\,dx}{\sqrt{2x^6+3x^3+1}}$.

6. $\displaystyle\int\frac{dx}{\sqrt{-3x^2-6x-2}}$.

7. $\displaystyle\int\frac{dx}{\sqrt{-27+10x+5x^2}}$.

8. $\displaystyle\int\frac{\sqrt{-m^2x+mn}}{(mx-n)\sqrt{mx+n}}\,dx$.

9. $\displaystyle\int\frac{\sqrt{2x-x^2}(\sqrt{2x-x^2}+g)+h}{34x-17x^2}\,dx$.

10. $\displaystyle\int\frac{\sqrt{2x-3}(\sqrt{2-3x}+\sqrt{2x+3})}{(2x-3)\sqrt{6-5x-6x^2}}\,dx$.

11. $\displaystyle\int\frac{(-2x+4)\,dx}{\sqrt{x^2+2x+4}}$.

12. $\displaystyle\int\frac{(x+1)\sqrt{x-2}}{\sqrt{3-x}}\,dx$.

13. $\displaystyle\int(x-1)^{\frac{3}{2}}(x+1)^{-\frac{1}{2}}\,dx$.

14. $\displaystyle\int\frac{dx}{(x+1)\sqrt{x^2+1}}$.

15. $\displaystyle\int\frac{dx}{(2x+3)\sqrt{x^2-2}}$.

16. $\displaystyle\int\frac{(x-4)\,dx}{(x+3)\sqrt{x^2+4x+5}}$.

17. $\displaystyle\int\frac{dx}{(x^2-x)\sqrt{1+2x-x^2}}$.

(Separate $\dfrac{1}{x^2-x}$ into partial fractions.)

18. $\displaystyle\int\frac{x\,dx}{(x^2-2)\sqrt{x^2-3}}$.

19. $\displaystyle\int\frac{(x^2+3x+5)\,dx}{(x+1)\sqrt{x^2+1}}$.

20. $\displaystyle\int\frac{(x^2+2x+3)\,dx}{(x^2+2x-3)\sqrt{1-x^2}}$.

21. $\displaystyle\int\sqrt{b^2+a^2x^2}(\sqrt{b^2+a^2x^2}+ax)^2\,dx$.

22. $\displaystyle\int\frac{dx}{(a^2-x^2)^2\sqrt{a^2-x^2}}$.

23. $\displaystyle\int\frac{x^6\,dx}{\sqrt{a^2-x^2}}$.

24. $\displaystyle\int\frac{x\,dx}{\sqrt{a^2-x^2}}$.

25. $\displaystyle\int\frac{x^5\,dx}{\sqrt{a^2-x^2}}$.

26. $\displaystyle\int\frac{dx}{x\sqrt{a^2-x^2}}$.

27. $\displaystyle\int\frac{\sqrt{a^2-x^2}}{x}\,dx$.

28. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx$.

29. $\displaystyle\int x^4\sqrt{a^2-x^2}\,dx.$

30. $\displaystyle\int\frac{dx}{x^2\sqrt{x^2\pm a^2}}.$

31. $\displaystyle\int\frac{dx}{x\sqrt{x^2+a^2}}.$

32. $\displaystyle\int x^2\sqrt{x^2\pm a^2}\,dx.$

33. $\displaystyle\int\frac{dx}{(x^2\pm a^2)^{\frac{3}{2}}}.$

34. $\displaystyle\int(x^2\pm a^2)^{\frac{3}{2}}\,dx.$

35. $\displaystyle\int\frac{x\,dx}{\sqrt{x^2\pm a^2}}.$

36. $\displaystyle\int\frac{x\,dx}{\sqrt{2\,ax-x^2}}.$

37. $\displaystyle\int\frac{dx}{x\sqrt{2\,ax-x^2}}.$

38. $\displaystyle\int\frac{x^3\,dx}{\sqrt{2\,ax-x^2}}.$

39. $\displaystyle\int\sqrt{2\,ax-x^2}\,dx.$

40. $\displaystyle\int\frac{dx}{(x^2+a^2)^3}.$

41. $\displaystyle\int\frac{x\,dx}{(x^4+a^4)^2}.$

42. $\displaystyle\int\frac{x^2\,dx}{(x^3-a^3)^2}.$

43. $\displaystyle\int\frac{dx}{(x^2-2\,x+3)^2}.$

44. $\displaystyle\int\frac{dx}{(x-1)^2\,(x+1)^2}.$

45. $\displaystyle\int\frac{\sqrt{1+x+x^2}}{1+x}\,dx.$

46. $\displaystyle\int\frac{\sqrt{x}\,dx}{x^3+6\,x^2+11\,x+6}.$

47. $\displaystyle\int\frac{2\,abx\,dx}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}}.$

48. $\displaystyle\int_0^{2a}\sqrt{2\,ax-x^2}\,\text{vers}^{-1}\frac{x}{a}\,dx.$

49. $\displaystyle\int\frac{-x^4+x^3+3\,x+9}{x^6+9\,x^4+27\,x^2+27}\,dx.$

50. $\displaystyle\int\frac{x^2+5}{\sqrt{x^2+2}}\,dx.$

$$\left(\frac{x^2+5}{\sqrt{x^2+2}}=\sqrt{x^2+2}+\frac{3}{\sqrt{x^2+2}}.\right)$$

51. $\displaystyle\int\frac{x^2+2\,x+1}{\sqrt{4\,x^2+4\,x+3}}\,dx.$

52. $\displaystyle\int\frac{x^2+x+1}{\sqrt{x^2+2\,x+3}}\,dx.$

53. $\displaystyle\int\frac{x^2-x-3}{\sqrt{-3+12\,x-9\,x^2}}\,dx.$

54. $\displaystyle\int\sqrt{lx^2+mx+n}\,dx,$ l negative.

55. $\displaystyle\int\frac{dx}{(x^2+a^2)\sqrt{x^2-a^2}}.$

56. $\displaystyle\int\frac{(x+1)\,dx}{(2\,x^2-2\,x+\frac{1}{2})\sqrt{3\,x^2-2\,x+1}}.$

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CHAPTER VII

INTEGRATION OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

Integration by parts and the use of substitutions will be found very helpful in obtaining the integrals of trigonometric and exponential functions.

46. $\int \sin^n x\, dx$, $\int \cos^n x\, dx$, n being an integer.

(a) If n be a positive odd integer, say $2m+1$,

$$\int \sin^n x\, dx = \int \sin^{2m+1} x\, dx = \int \sin^{2m} x \sin x\, dx$$

$$= -\int (1-\cos^2 x)^m d(\cos x).$$

The binomial in the latter form can be expanded, and the integral found term by term.

Ex. 1. $\int \sin^3 x\, dx = -\int (1-\cos^2 x) d(\cos x) = -\cos x + \frac{1}{3}\cos^3 x + c.$

Similarly,

$$\int \cos^{2m+1} x\, dx = \int (1-\sin^2 x)^m d(\sin x) = \sin x - \frac{m}{3}\sin^3 x + \cdots.$$

Ex. 2. $\int \cos^5 x\, dx = \int (1-\sin^2 x)^2 d(\sin x) = \int (1-2\sin^2 x + \sin^4 x) d(\sin x)$

$$= \sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x + c.$$

(b) If n be any positive integer, integrate $\sin^n x\, dx$ by parts, putting

$$u = \sin^{n-1} x, \quad dv = \sin x\, dx.$$

Then, $\quad du = (n-1)\sin^{n-2}x\cos x\,dx,\ v = -\cos x;$

and
$$\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$
$$-(n-1)\int \sin^n x\,dx.$$

By transposing the last term to the first member, combining, and dividing through the equation by n, the result is

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx. \qquad [A]$$

The result A can be used as a *reduction formula.* Successive applications of it leads to $\int dx$ or $\int \sin x\,dx$ according as n is even or odd.

Ex. 3. $\int \sin^3 x\,dx = -\frac{\sin^2 x\cos x}{3} + \frac{2}{3}\int \sin x\,dx$

$$= -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + c.$$

This result may be compared with that of Ex. 1.

On integrating $\cos^n x\,dx$ by parts, putting $u = \cos^{n-1}x$, $dv = \cos x\,dx$, the following reduction formula is obtained,

$$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx. \qquad [B]$$

The deduction of B is left as an exercise.

(c) Suppose that n is a negative integer.

The value of $\int \sin^{n-2}x\,dx$ in A is

$$\int \sin^{n-2}x\,dx = \frac{\sin^{n-1}x\cos x}{n-1} + \frac{n}{n-1}\int \sin^n x\,dx.$$

On changing n into $n+2$ this becomes

$$\int \sin^n x\,dx = \frac{\sin^{n+1} x \cos x}{n+1} + \frac{n+2}{n+1}\int \sin^{n+2} x\,dx. \qquad [C]$$

This can be used as a formula of reduction when n is a negative integer.

Ex. 4. $\int \frac{dx}{\sin^3 x} = \int (\sin x)^{-3}\,dx = -\tfrac{1}{2}\cos x(\sin x)^{-2} + \int (\sin x)^{-1}\,dx$

$$= -\frac{\cos x}{2\sin^2 x} + \int \csc x\,dx$$

$$= -\frac{1}{2}\cot x \csc x + \log\tan\frac{x}{2} + c.$$

Similarly, on solving for $\int \cos^{n-2} x\,dx$ in B, and then changing n into $n+2$, there results,

$$\int \cos^n x\,dx = -\frac{\sin x \cos^{n+1} x}{n+1} + \frac{n+2}{n+1}\int \cos^{n+2} x\,dx. \qquad [D]$$

This is a formula of reduction for $\int \cos^n x\,dx$ when n is negative.

It is advisable to remember the method of deriving A, B, C, D, so that they may be readily obtained when necessary.

Ex. 5. (*a*) $\int \cos^7 x\,dx$, (*b*) $\int \sin^5 x\,dx$, (*c*) $\int \sin^7 x\,dx$.

Ex. 6. (*a*) $\int \sin^4 x\,dx$, (*b*) $\int \sin^6 x\,dx$, (*c*) $\int \sin^8 x\,dx$.

Ex. 7. (*a*) $\int \cos^4 x\,dx$, (*b*) $\int \cos^6 x\,dx$, (*c*) $\int \cos^8 x\,dx$.

Ex. 8. (*a*) $\int \frac{dx}{\sin^5 x}$, (*b*) $\int \frac{dx}{\cos^3 x}$, (*c*) $\int \frac{dx}{\cos^4 x}$, (*d*) $\int \frac{dx}{\cos^5 x}$,

(*e*) $\int \frac{dx}{\cos^6 x}$, (*f*) $\int \frac{dx}{\sin^4 x}$, (*g*) $\int \frac{dx}{\sin^6 x}$.

Ex. 9. Show that $\int_0^{\frac{\pi}{2}} \sin^{2m} dx = \frac{1\cdot 3\cdot 5 \cdots (2m-1)}{2\cdot 4\cdot 6 \cdots 2m}\cdot\frac{\pi}{2}$.

Ex. 10. Show that $\int_0^{\frac{\pi}{2}} \sin^{2m+1} x\,dx = \frac{2\cdot 4\cdot 6 \cdots 2m}{3\cdot 5\cdot 7 \cdots 2m+1}$.

47. Algebraic transformations. The trigonometric integral $\int \sin^n x\,dx$ can be put into an algebraic form. For, if

$$\sin x = z,$$

then
$$\cos x\,dx = dz,$$

and
$$dx = \frac{dz}{\cos x} = \frac{dz}{\sqrt{1-z^2}};$$

and hence,
$$\int \sin^n x\,dx = \int \frac{z^n\,dz}{\sqrt{1-z^2}}.$$

The second member has a form which has been discussed in Chapter VI.

If the substitution
$$\cos x = z$$

be made,
$$\int \sin^n x\,dx = -\int (1-z^2)^{\frac{n-1}{2}}\,dz.$$

This form can be integrated by methods already explained. These substitutions may also be employed in the case of $\int \cos^n x\,dx$.

Ex. 1. $\int \sin^3 x\,dx$. (Compare with Ex. 3, Art. 46.)

On putting

$$z = \sin x,\ \int \sin^3 x\,dx = \int \frac{z^3\,dz}{\sqrt{1-z^2}} = \frac{-z^2\sqrt{1-z^2}}{3} - \frac{2}{3}\sqrt{1-z^2} + c$$

$$= -\tfrac{1}{3}\sin^2 x\cos x - \tfrac{2}{3}\cos x + c.$$

Ex. 2. Solve Exs. 2, 4, 6 (*a*), 7 (*a*), 8 (*b*) of Art. 46 by algebraic substitution.

48. $\int \sec^n x\,dx$, $\int \operatorname{cosec}^n x\,dx$.

(a) Since $\sec x = \frac{1}{\cos x}$, and $\operatorname{cosec} x = \frac{1}{\sin x}$, $\int \sec^n x\,dx$ and $\int \operatorname{cosec}^n x\,dx$ can be reduced to the forms considered in Arts. 46, 47.

Ex. 1. (*a*) $\int \sec^4 x\, dx$, (*b*) $\int \operatorname{cosec}^6 x\, dx$, (*c*) $\int \frac{dx}{\sec^5 x}$, (*d*) $\int \frac{dx}{\operatorname{cosec}^4 x}$. [See Exs. 8 (*c*), 8 (*g*), 8 (*d*), 6 (*a*), (Art. 46).]

Ex. 2. Find $\int \frac{dx}{(x^2 + a^2)^k}$, assuming $x = a \tan \theta$. (See Art. 39, and Ex. 2, Art. 45.)

(**b**) If n is an even positive integer, another method may be employed.

Since $\sec^2 x = 1 + \tan^2 x$, and $d(\tan x) = \sec^2 x\, dx$,

$$\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$$

$$= \int (1 + \tan^2 x)^{\frac{n-2}{2}} d(\tan x).$$

The binomial under the sign of integration can be expanded in a finite number of terms since n is even, and accordingly $\frac{n-2}{2}$ is an integer.

In a similar way it can be shown that

$$\int \operatorname{cosec}^n x\, dx = -\int (1 + \cot^2 x)^{\frac{n-2}{2}} d(\cot x).$$

Ex. 3. $\int \sec^6 x\, dx = \int (1 + \tan^2 x)^2 d(\tan x)$;

$$= \tan x + \tfrac{2}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x + c.$$

(Compare Ex. 8 (*e*), Art. 46.)

Ex. 4. (*a*) $\int \sec^4 x\, dx$; (*b*) $\int \operatorname{cosec}^4 x\, dx$;

(*c*) $\int \operatorname{cosec}^6 x\, dx$; (*d*) $\int \sec^4 \frac{x}{3}\, dx$.

[Compare the results with those of Exs. 8 (*c*), 8 (*f*), 8 (*g*), Art. 46.]

(**c**) If n is any positive integer greater than 2, the method of integration by parts can be used.

On putting $\sec^{n-2} x = u$, $\sec^2 x\, dx = dv$,

it follows that $du = (n - 2) \sec^{n-2} x \tan x\, dx$, $v = \tan x$.

Hence, $\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$

$$= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx.$$

On substituting $\sec^2 x - 1$ for $\tan^2 x$ in the last term, and solving for $\int \sec^n x\, dx$, there is obtained

$$\int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx. \qquad [A]$$

This formula of reduction leads to $\int \sec x\, dx$ when n is odd, and to $\int dx$ when n is even.

Similarly, integration by parts will give

$$\int \operatorname{cosec}^n x\, dx = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\, dx, \quad [B]$$

which, on repeated applications, leads to $\int \operatorname{cosec} x\, dx$ or to $\int dx$, according as n is odd or even.

Ex. 5. (*a*) $\int \sec^3 x\, dx$; (*b*) $\int \sec^5 x\, dx$; (*c*) $\int \operatorname{cosec}^3 x\, dx$;

(*d*) $\int \operatorname{cosec}^5 x\, dx$; (*e*) $\int \sec^3 \frac{2}{3} x\, dx$.

[Compare the results with those of Exs. 8 (*b*), 8 (*d*), 4, 8 (*a*), Art. 46.]

Ex. 6. (*a*) $\int \sec^4 x\, dx$; (*b*) $\int \operatorname{cosec}^4 x\, dx$. [Compare Ex. 4 (*a*), (*b*).]

(d) Transformation to an algebraic form.

If $\tan x = z$, $x = \tan^{-1} z$, $dx = \frac{dz}{1+z^2}$, $\sec^2 x = 1 + z^2$;

and hence, $$\int \sec^n x\, dx = \int (1+z^2)^{\frac{n-2}{2}} dz.$$

Also, if $\sec x = z$,

it follows that $$dx = \frac{dz}{z\sqrt{z^2-1}};$$

and hence, $$\int \sec^n x\,dx = \int \frac{z^{n-1}}{\sqrt{z^2-1}}\,dz.$$

In like manner, the substitutions $z = \cot x$, $z = \operatorname{cosec} x$, will reduce $\int \operatorname{cosec}^n x\,dx$ to an algebraic form.

Ex. 7. Solve Exs. 3 (*a*), 4 (*a*), 4 (*b*), 1 (*d*) above, by algebraic substitution.

49. $\int \tan^n x\,dx$, $\int \cot^n x\,dx$.

(a) Let n be a positive integer.

Then, $$\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$$
$$= \int \tan^{n-2} x (\sec^2 x - 1)\,dx$$
$$= \int \tan^{n-2} x\,d(\tan x) - \int \tan^{n-2} x\,dx$$
$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx. \qquad [A]$$

This reduction formula leads to $\int dx$ or to $\int \tan x\,dx$, according as n is even or odd.

Ex. 1. $\int \tan^3 x\,dx = \int \tan x (\sec^2 x - 1)\,dx = \int \tan x\,d(\tan x) - \int \tan x\,dx$
$$= \tfrac{1}{2}\tan^2 x - \log \sec x + c.$$

In like manner,

$$\int \cot^n x\,dx = \int \cot^{n-2} x \cot^2 x\,dx = \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\,dx$$
$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx, \qquad [B]$$

another reduction formula.

(b) If n is a negative integer, say $-m$,

$$\int \tan^n x\,dx = \int \cot^m x\,dx, \text{ and } \int \cot^n x\,dx = \int \tan^m x\,dx.$$

Hence, this case reduces to the preceding.

Ex. 2. (*a*) $\int \tan^4 x\,dx$, (*b*) $\int \tan^5 x\,dx$, (*c*) $\int \tan^6 x\,dx$, (*d*) $\int \cot^3 x\,dx$, (*e*) $\int \cot^4 x\,dx$, (*f*) $\int \cot^5 x\,dx$.

(c) Transformation to an algebraic form.

If $$\tan x = z,\ dx = \frac{dz}{1+z^2};$$

and hence, $$\int \tan^n x\,dx = \int \frac{z^n\,dz}{1+z^2}.$$

Again, if $$\sec x = z,\ dx = \frac{dz}{z\sqrt{z^2-1}};$$

and hence, $$\int \tan^n x\,dx = \int \frac{(z^2-1)^{\frac{n-1}{2}}}{2}\,dz.$$

Similarly, $\int \cot^n x\,dx$ can be changed into algebraic forms by the substitutions,

$$z = \cot x, \text{ and } z = \operatorname{cosec} x.$$

Ex. 3. Solve Ex. 1, 2 (*a*), 2 (*d*) above, by algebraic substitution.

50. $\int \sin^m x \cos^n x\,dx$.

(a) When either m or n is a positive odd integer, no matter what the other may be, the form $\sin^m x \cos^n x\,dx$ can be integrated very easily. For, it is then reducible to a sum of integrals of the form $\int \sin^p x\,d(\sin x)$, or of the form $\int \cos^q x\,d(\cos x)$. This is illustrated by the following example.

Ex. 1. $\sin^{\frac{3}{7}} x \cos^3 x\, dx = \int \sin^{\frac{3}{7}} x\,(1 - \sin^2 x)\, d\,(\sin x)$

$$= \tfrac{7}{10} \sin^{\frac{10}{7}} x - \tfrac{7}{24} \sin^{\frac{24}{7}} x + c.$$

Ex. 2. $\int \sin^{\frac{2}{3}} x \cos^5 x\, dx.$

Ex. 3. $\int \cos^{\frac{3}{4}} x \sin^3 x\, dx.$

Ex. 4. $\int \frac{\sin^5 x}{\sqrt{\cos x}}\, dx.$

Ex. 5. $\int \frac{\cos^5 x}{\sqrt[3]{\sin x}}\, dx.$

(b) When $m + n$ is a negative even integer, say $-2p$, $\sin^m x \cos^n x\, dx$ can be integrated by means of the substitution $\tan x = t$.

If $\tan x = t,\ dx = \frac{dt}{1 + t^2},\ \sin x = \frac{t}{\sqrt{1 + t^2}},\ \cos x = \frac{1}{\sqrt{1 + t^2}};$

and hence, $\int \sin^m x \cos^n x\, dx = \int \frac{t^m\, dt}{(1 + t^2)^{\frac{m+n}{2}+1}} = \int t^m (1 + t^2)^{p-1}\, dt.$

The last form is readily integrable. Sometimes the substitution $\cot x = t$ is better than the substitution $\tan x = t$ for obtaining a simple integrable form.

$\int \frac{\sin^2 x}{\cos^6 x}\, dx.$

If $\tan x = t$, $\int \frac{\sin^2 x}{\cos^6 x}\, dx = \int t^2 (1 + t^2)\, dt = \frac{1}{3} t^3 + \frac{1}{5} t^5 + c$

$$= \tfrac{1}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x + c.$$

The actual substitution of the new variable may often be conveniently omitted; for instance,

$$\int \frac{\sin^2 x}{\cos^6 x}\, dx = \int \tan^2 x \sec^4 x\, dx = \int \tan^2 x\,(1 + \tan^2 x)\, d(\tan x)$$
$$= \tfrac{1}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x + c.$$

Ex. 6. $\int \frac{\sin^3 x}{\cos^7 x}\, dx.$

Ex. 7. $\int \frac{\cos^4 x}{\sin^8 x}\, dx.$

(c) Transformation to an algebraic form.

If $\qquad \sin x = z, \quad \cos x = \sqrt{1 - z^2}, \quad dx = \frac{dz}{\sqrt{1 - z^2}};$

and hence, $\qquad \int \sin^m x \cos^n x\, dx = \int z^m (1 - z^2)^{\frac{n-1}{2}}\, dz.$

In like manner the substitution $z = \cos x$ will give

$$\int \sin^m x \cos^n x\, dx = \int z^n (1 - z^2)^{\frac{m-1}{2}}\, dz.$$

The substitution $\tan x = t$ leads to a simple form when $m + n$ is a negative even integer. This has been shown above. See Ex. 6.

51. Integration of $\sin^m x \cos^n x\, dx$: (*a*) by the method of undetermined coefficients; (*b*) by means of reduction formulæ.

(*a*) Rule. Put $\int \sin^m x \cos^n x\, dx$ equal to a constant times one of the four integrals,

$$\int \sin^{m-2} x \cos^n x\, dx, \quad \int \sin^m x \cos^{n-2} x\, dx,$$

$$\int \sin^{m+2} x \cos^n x\, dx, \quad \int \sin^m x \cos^{n+2} x\, dx,$$

plus a constant times $\sin^{p+1} x \cos^{q+1} x$, in which p, q are the lowest indices of $\sin x$ and $\cos x$ in the two expressions under the integration sign. Then determine the values of the two constant coefficients by differentiating, simplifying, and equating coefficients of like terms. For instance, using the first of the four integrals referred to in the rule,

$$\int \sin^m x \cos^n x\, dx \equiv A \sin^{m-1} x \cos^{n+1} x + B \int \sin^{m-2} x \cos^n x\, dx.$$

Here the lowest indices of $\sin x$, $\cos x$, under the sign $\int$, are $m - 2$, n; and from them the term $A \sin^{m-1} x \cos^{n+1} x$ is formed by the rule. On finding the differential coefficients of both members of this equation, there is obtained

$$\sin^m x \cos^n x \equiv (m-1) A \sin^{m-2} x \cos^{n+2} x - (n+1) A \sin^m x \cos^n x$$
$$+ B \sin^{m-2} x \cos^n x.$$

Division of both terms by $\sin^{m-2} x \cos^n x$ gives

$$\sin^2 x \equiv (m-1) A (1 - \sin^2 x) - (n+1) A \sin^2 x + B.$$

On equating the coefficients of like terms in both members,

$$-(m+n)A = 1,$$

$$(m-1)A + B = 0;$$

whence, $$A = -\frac{1}{m+n}, \quad B = \frac{m-1}{m+n}.$$

Hence,

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x \, dx. \quad \text{[A]}$$

In a similar way, by connecting $\int \sin^m x \cos^n x \, dx$ with the other integrals mentioned in the rule, the following reduction formulæ are obtained:

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x \, dx. \quad \text{[B]}$$

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x \, dx. \quad \text{[C]}$$

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{n+m+2}{n+1}\int \sin^m x \cos^{n+2} x \, dx. \quad \text{[D]}$$

In each of the four integrals with which $\int \sin^m x \cos^n x \, dx$ may be connected by the rule, m or n is increased or diminished by 2. The numerical values of m, n will indicate the one of the four which is simpler than $\int \sin^m x \cos^n x \, dx$, and with which it may preferably be connected. A succession of steps like **A, B, C, D,** may be necessary.

In solving the problems of this and the following article, it may happen that the results will not agree in form with those given in the answers. An agreement can be made by using the trigonometric relations, $\sin^2 x + \cos^2 x = 1$, $\sec^2 x = 1 + \tan^2 x$, etc. Any apparent difference in results will be due to a difference in the methods of working the examples.

Ex. 1. $\int \sin^2 x \cos^2 x\, dx.$

Assume $$\int \sin^2 x \cos^2 x\, dx = A \sin x \cos^3 x + B \int \cos^2 x\, dx.$$

Differentiating, $$\sin^2 x \cos^2 x = A \cos^4 x - 3A \sin^2 x \cos^2 x + B \cos^2 x,$$

whence, dividing by $\cos^2 x$, $\sin^2 x = A(1 - \sin^2 x) - 3A \sin^2 x + B.$

Equating coefficients of like terms,

$$-4A = 1,$$
$$A + B = 0.$$

On solving these equations, $A = -\frac{1}{4}$, $B = \frac{1}{4}$.

Hence, $$\int \sin^2 x \cos^2 x\, dx = -\tfrac{1}{4} \sin x \cos^3 x + \tfrac{1}{4} \int \cos^2 x\, dx;$$

from this, by Art. 46, $$= -\tfrac{1}{4} \sin x \cos^3 x + \tfrac{1}{8} (\sin x \cos x + x) + c.$$

Equally well, $\int \sin^2 x \cos^2 x\, dx$ might have been connected with $\int \sin^2 x\, dx$. Also, $1 - \cos^2 x$ might have been substituted for $\sin^2 x$, or $1 - \sin^2 x$ for $\cos^2 x$, and the integral found by the method of Art. 46.

Ex. 2. $\int \sin^4 x \cos^2 x\, dx.$

Ex. 3. $\int \frac{dx}{\sin^4 x \cos^2 x}.$

Ex. 4. $\int \frac{\cos^4 x}{\sin^2 x} dx.$

Ex. 5. $\int \frac{\cos^4 x}{\sin^3 x} dx$

Ex. 6. Solve some of these examples by reducing them to an algebraic form, as described in Art. 50 (*c*).

(b) The results **A**, **B**, **C**, **D** can be used as formulæ of reduction. It is necessary only to substitute in them the proper values for m and n. It is not necessary to memorize these formulæ. The student should make himself familiar with the process of deriving these formulæ, so that he can readily obtain them when required.* The formulæ **A**, **B**, **C**, **D** of Art. 46 are special cases of **A**, **B**, **C**, **D** above. This will be apparent on putting m and n in turn equal to zero in the latter formulæ. Moreover, $\int \tan^n x\,dx$, $\int \cot^n x\,dx$, discussed in Art. 49, may be put in the forms $\int \sin^n x \cos^{-n} x\,dx$, $\int \cos^n x \sin^{-n} x\,dx$, and solved by the methods of this article. For the sake of practice in making the substitutions a few examples may be solved by means of the formulæ.

Ex. 7. $\int \sin^6 x \cos^4 x\,dx$.

By A, $$\int \sin^6 x \cos^4 x\,dx = -\frac{\sin^5 x \cos^5 x}{10} + \frac{5}{10}\int \sin^4 x \cos^4 x\,dx\,;$$

by A, $$\int \sin^4 x \cos^4 x\,dx = -\frac{\sin^3 x \cos^5 x}{8} + \frac{3}{8}\int \sin^2 x \cos^4 x\,dx\,;$$

by A, $$\int \sin^2 x \cos^4 x\,dx = -\frac{\sin x \cos^5 x}{6} + \frac{1}{6}\int \cos^4 x\,dx\,;$$

by C, $$\int \cos^4 x\,dx = \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\,dx\,;$$

by C, $$\int \cos^2 x\,dx = \frac{\sin x \cos x}{2} + \frac{1}{2}\int dx = \frac{\sin x \cos x}{2} + \tfrac{1}{2}x + c.$$

The combination of the results gives

$$\int \sin^6 x \cos^4 x\,dx = -\tfrac{1}{10}\sin^5 x \cos^5 x - \tfrac{1}{16}\sin^3 x \cos^5 x - \tfrac{1}{32}\sin x \cos^5 x$$
$$+ \tfrac{1}{128}\sin x \cos^3 x + \tfrac{3}{256}\sin x \cos x + \tfrac{3}{256}x + c.$$

The formulæ might have been applied in other orders, for example $CACAA$, $CCAAA$, etc.

Ex. 8. Solve Exs. 2, 3, 4, 5 by means of the reduction formulæ.

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* Another method of deriving these formulæ, namely, by integrating by parts, is given in Note C, Appendix. Other formulæ of reduction can be obtained by connecting

$$\int \sin^m x \cos^n x\,dx \text{ with } \int \sin^{m-2} x \cos^{n+2} x\,dx,\ \int \sin^{m+2} x \cos^{n-2} x\,dx$$

in the manner described above. The formulæ for these cases may be derived as an exercise. (See Edwards, *Integral Calculus*, Art. 83.)

52. $\int \tan^m x \sec^n x\, dx$, $\int \cot^m x \operatorname{cosec}^n x\, dx$.

(a) Reduction to the form $\int \sin^M x \cos^N x\, dx$. This may be done by the substitutions

$$\tan x = \frac{\sin x}{\cos x},\quad \sec x = \frac{1}{\cos x},\quad \cot x = \frac{\cos x}{\sin x},\quad \operatorname{cosec} x = \frac{1}{\sin x};$$

and the integration can then be performed by one of the methods of the last two articles.

(b) Reduction to an algebraic form.

If $\tan x = z$, $\quad \int \tan^m x \sec^n x\, dx = \int z^m (1+z^2)^{\frac{n-2}{2}}\, dz.$

This is almost immediately integrable if n is a positive even integer. In this case, $\int \tan^m x \sec^n x\, dx$ can be reduced to integrals of the form $\int \tan^p x\, d(\tan x)$. (See Ex. 1, below.)

If $\sec x = z$, $\quad \int \tan^m x \sec^n x\, dx = \int z^{n-1} (z^2-1)^{\frac{m-1}{2}}\, dz.$

This is almost immediately integrable if m is a positive odd integer. In this case, $\int \tan^m x \sec^n x\, dx$ can be reduced to integrals of the form $\int \sec^q x\, d(\sec x)$. (See Ex. 1, below.)

The form $\int \cot^m x \operatorname{cosec}^n x\, dx$ may be treated in a similar manner.

Ex. 1.
$$\begin{aligned}\int \tan^3 x \sec^4 x\, dx &= \int \tan^3 x \sec^2 x \sec^2 x\, dx \\ &= \int \tan^3 x (\tan^2 x + 1)\, d(\tan x) \\ &= \tfrac{1}{6}\tan^6 x + \tfrac{1}{4}\tan^4 x + c.\end{aligned}$$

Or,
$$\begin{aligned}\int \tan^3 x \sec^4 x\, dx &= \int \tan^2 x \sec^3 x \sec x \tan x\, dx \\ &= \int (\sec^2 x - 1)\sec^3 x\, d(\sec x) \\ &= \tfrac{1}{6}\sec^6 x - \tfrac{1}{4}\sec^4 x + c.\end{aligned}$$

Ex. 2. $\int \frac{\sec^6 x\,dx}{\tan^3 x}.$ **Ex. 5.** $\int \cot^5 x \operatorname{cosec}^5 x\,dx.$

Ex. 3. $\int \tan^{\frac{2}{3}} x \sec^4 x\,dx.$ **Ex. 6.** $\int \tan^7 x \sec^3 x\,dx.$

Ex. 4. $\int \cot^3 x \operatorname{cosec}^4 x\,dx.$ **Ex. 7.** $\int \tan^5 x \sec^{\frac{3}{4}} x\,dx.$

Ex. 8. Solve some of these examples by using algebraic transformations.

53. Use of multiple angles. When m and n are positive and one of them is odd, the first method of integration shown in Art. 50 can be employed in the case of $\int \sin^m x \cos^n x\,dx$. When m and n are positive and both even, the use of multiple angles will aid the process of integration. The trigonometric substitutions that can be employed for this purpose are:

$$\sin x \cos x = \tfrac{1}{2} \sin 2x,$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

Ex. 1. $\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx$

$$= \tfrac{1}{8}\int (1 - \cos 4x)\,dx$$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + c.$$ (Compare Ex. 1, Art. 51.)

Ex. 2. $\int \sin^4 x\,dx.$ (See Ex. 6 (*a*), Art. 46.)

Ex. 3. $\int \cos^4 x\,dx.$ (See Ex. 7 (*a*), Art. 46.)

Ex. 4. $\int \sin^6 x\,dx.$ (See Ex. 6 (*b*), Art. 46.)

Ex. 5. $\int \cos^6 x\,dx.$ (See Ex. 7 (*b*), Art. 46.)

Ex. 6. $\int \sin^4 x \cos^2 x\,dx.$ (See Ex. 2, Art. 51.)

Ex. 7. $\int \sin^4 x \cos^4 x\,dx.$

54. $\displaystyle\int\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}.$

On denoting the integral by I, and dividing the numerator and denominator by $\cos^2 x$,

$$I=\int\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}=\int\frac{d(\tan x)}{a^2+b^2\tan^2 x}.$$

On substituting u for $\tan x$, integrating, and replacing u by $\tan x$, there is obtained,

$$I=\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right).$$

55. $\displaystyle\int\frac{dx}{a+b\cos x},\ \int\frac{dx}{a+b\sin x}.$

Since $\qquad \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$, and $\cos^2\frac{x}{2}+\sin^2\frac{x}{2}=1$,

$$\int\frac{dx}{a+b\cos x}=\int\frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}. \qquad (1)$$

On dividing numerator and denominator in the second member by $\cos^2\frac{x}{2}$ and reducing,

$$\int\frac{dx}{a+b\cos x}=\frac{1}{a-b}\int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+\frac{a+b}{a-b}}$$

$$=\frac{2}{a-b}\int\frac{d\left(\tan\frac{x}{2}\right)}{\tan^2\frac{x}{2}+\frac{a+b}{a-b}}.$$

The second member is of the form $\displaystyle\int\frac{dz}{z^2+c^2}$, or $\displaystyle\int\frac{dz}{z^2-c^2}$, according as a is greater than b, or less than b. Hence,

if $a>b$, $\displaystyle\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$;

$$\text{if } a < b,\quad \int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b+a} + \sqrt{b-a}\tan\dfrac{x}{2}}{\sqrt{b+a} - \sqrt{b-a}\tan\dfrac{x}{2}}$$

$$= \frac{2}{\sqrt{b^2 - a^2}} \tanh^{-1}\left(\sqrt{\frac{b-a}{b+a}}\tan\frac{x}{2}\right).$$

On introducing the half-angle as before, and dividing numerator and denominator by $\cos^2\frac{x}{2}$, as in the case just considered, it will be found that,

$$\int \frac{dx}{a + b\sin x} = \int \frac{dx}{a\left(\cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2}\right) + 2\,b\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$$

$$= \int \frac{\sec^2\dfrac{x}{2}\,dx}{a + 2\,b\tan\dfrac{x}{2} + a\tan^2\dfrac{x}{2}}$$

$$= \frac{2}{a}\int \frac{d\left(\tan\dfrac{x}{2} + \dfrac{b}{a}\right)}{\left(\tan\dfrac{x}{2} + \dfrac{b}{a}\right)^2 + \dfrac{a^2 - b^2}{a^2}}.$$

This is in the form $\int \frac{dz}{z^2 + c^2}$, or $\int \frac{dz}{z^2 - c^2}$, according as a is greater or less than b. Hence,

$$\text{if } a > b,\quad \int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\frac{a\tan\dfrac{x}{2} + b}{\sqrt{a^2 + b^2}}\right);$$

$$\text{if } a < b,\quad \int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{a\tan\dfrac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\dfrac{x}{2} + b + \sqrt{b^2 - a^2}};$$

$$= -\frac{2}{\sqrt{b^2 - a^2}} \coth^{-1}\left(\frac{a\tan\dfrac{x}{2} + b}{\sqrt{b^2 - a^2}}\right).$$

In working the examples it is preferable to follow the method employed above, and not to use the results that have just been found as formulæ for substitution.

Ex. 1. $\int \frac{dx}{3 - 2\sin x}$.

Ex. 2. $\int \frac{dx}{2 + 3\sin 2x}$.

Ex. 3. $\int \frac{dx}{5 + 3\cos x}$.

Ex. 4. $\int \frac{dx}{5 - 3\cos x}$.

Ex. 5. $\int \frac{dx}{4 + 5\cos 2x}$.

Ex. 6. $\int \frac{dx}{4 + 5\sin 2x}$.

Ex. 7. When $a = b$ in this article, find the integrals.

56. $\int e^{ax} \sin nx\, dx, \quad \int e^{ax} \cos nx\, dx.$

On integrating $e^{ax}\sin nx\, dx$ by parts, first taking $e^{ax}\,dx$ for dv, and then taking $\sin nx\, dx$ for dv, there are obtained

$$\int e^{ax} \sin nx\, dx = \frac{e^{ax}\sin nx}{a} - \frac{n}{a}\int e^{ax}\cos nx\, dx, \tag{1}$$

$$\int e^{ax} \sin nx\, dx = -\frac{e^{ax}\cos nx}{n} + \frac{a}{n}\int e^{ax}\cos nx\, dx. \tag{2}$$

The integral in the second members of (1), (2) can be eliminated by multiplying the members of (1) by $\frac{a}{n}$ and the members of (2) by $\frac{n}{a}$, and adding the results. When this is done it will be found that

$$\int e^{ax} \sin nx\, dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2 + n^2}. \tag{3}$$

Similarly, on integrating $e^{ax}\cos nx\, dx$ by parts, first taking $e^{ax}\,dx$ for dv, and then taking $\cos nx\, dx$ for dv, and eliminating $\int e^{ax}\sin nx\, dx$ which has thus been introduced, there will be obtained

$$\int e^{ax} \cos nx\, dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2 + n^2}, \tag{4}$$

The result (4) can be obtained by eliminating $\int e^{ax} \sin nx\, dx$ from (1) and (2). It can be deduced also by substituting the result (3) in (1) or in (2). It is, however, preferable to deduce it directly by integrating by parts.

As in Art. 55 the student is advised to work the examples by the method followed above, and not to use (3) and (4) as formulæ for substitution.

Ex. 1. $\int e^x \sin x\, dx.$

Ex. 2. $\int e^x \cos x\, dx.$

Ex. 3. $\int e^{2x} \cos 3x\, dx.$

Ex. 4. $\int \frac{\cos 2x}{e^{3x}} dx.$

Ex. 5. $\int \frac{\sin x}{e^x} dx.$

Ex. 6. $\int e^x \cos^2 x\, dx.$

57. $\int \sin mx \cos nx\, dx$, $\int \cos mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$.

Since, by trigonometry,

$$\sin mx \cos nx = \tfrac{1}{2} \sin (m+n)x + \tfrac{1}{2} \sin (m-n)x,$$

$$\int \sin mx \cos nx\, dx = -\frac{\cos (m+n)x}{2(m+n)} - \frac{\cos (m-n)x}{2(m-n)}.$$

In a similar way it can be shown that,

$$\int \cos mx \cos nx\, dx = \frac{\sin (m+n)x}{2(m+n)} + \frac{\sin (m-n)x}{2(m-n)},$$

$$\int \sin mx \sin nx\, dx = -\frac{\sin (m+n)x}{2(m+n)} + \frac{\sin (m-n)x}{2(m-n)}.$$

Ex. 1. $\int \cos 3x \sin 5x\, dx$

Ex. 2. $\int \cos 4x \cos 7x\, dx.$

Ex. 3. $\int \sin 5x \sin 6x\, dx.$

Ex. 4. $\int \cos 3x \cos \frac{4}{3}x\, dx.$

Ex. 5. $\int \cos \frac{3}{4}x \sin \frac{1}{4}x\, dx.$

Ex. 6. $\int \sin \frac{7}{10}x \sin \frac{3}{10}x\, dx.$

CHAPTER VIII

SUCCESSIVE INTEGRATION. MULTIPLE INTEGRALS

58. Successive integration. It has been seen in the differential calculus that *successive differentiation* with respect to x is sometimes required in the case of functions of the form

$$u = f(x);$$

and that successive differentiation with respect to both x and y may be required in the case of functions of the form

$$u = f(x, y).$$

On the other hand, the reverse process called *successive integration* is sometimes necessary. This chapter will be concerned with describing the notation that is used in "multiple integration," as it is often termed; and it will show, by examples, how successive integration is introduced and conducted. Arts. 61, 62, 63, contain applications of multiple integration to the measurement of areas in rectangular coördinates, and of volumes in rectangular and polar coördinates. Plane areas in polar coördinates and curvilinear surfaces will be found by means of multiple integration in Arts. 67, 75.

59. Successive integration with respect to a single independent variable.

Suppose that
$$f_1(x) = \int f(x)\,dx, \tag{1}$$

$$f_2(x) = \int f_1(x)\,dx, \tag{2}$$

and $$f_3(x) = \int f_2(x)\,dx. \tag{3}$$

Since $$f_2(x) = \int [f_1(x)]\,dx,$$

it follows from (1) that $$f_2(x) = \int \left[\int f(x)\,dx \right] dx; \tag{4}$$

and since $$f_3(x) = \int \{f_2(x)\}\,dx,$$

it follows from (4) that $$f_3(x) = \int \left\{ \int \left[\int f(x)\,dx \right] dx \right\} dx. \tag{5}$$

The second member of (4) is usually written in a contracted form, namely,

$$\int\int f(x)\,dx\,dx, \text{ or } \int\int f(x)\,dx^2, \tag{6}$$

in which dx^2 means $(dx)^2$, and not $d(x^2)$.

Similarly, the second member of (5) is usually written

$$\int\int\int f(x)\,dx\,dx\,dx, \quad \text{or } \int\int\int f(x)\,dx^3. \tag{7}$$

Integral (6) is called a double integral, and integral (7) is called a triple integral. In general, if an integral is evaluated by means of two or more successive integrations, it is called a *multiple integral.* If limits are assigned for each successive integration, the integral is definite; if limits are not assigned, it is indefinite.

Ex. 1. Determine the curve for every point of which the second differential coefficient of the ordinate with respect to the abscissa is 8.

The given condition is expressed by the equation

$$(1) \qquad \frac{d^2y}{dx^2} = 8.$$

This may be written $$\frac{d\left(\dfrac{dy}{dx}\right)}{dx} = 8;$$

whence, $$d\left(\frac{dy}{dx}\right) = 8\,dx.$$

Integrating, (2) $\frac{dy}{dx} = 8x + c,$

whence, $dy = (8x + c)dx.$

Integrating again, (3) $y = 4x^2 + cx + k.$

This is the equation of any parabola that has its axis parallel to the y-axis and drawn upwards, and its latus-rectum equal to 4. All such parabolas will be obtained by giving all possible values to c and k, the arbitrary constants of integration. Two further conditions will serve to make c and k definite. For instance, suppose that the tangent to the parabola at the point whose abscissa is 2, is parallel to the x-axis; and also that the parabola passes through the point (3, 5). By the former condition,

$$\frac{dy}{dx} = 0 \text{ when } x = 2;$$

and hence, by (2), $0 = 8 \cdot 2 + c,$

that is, $c = -16.$

Equation (3) then becomes $y = 4x^2 - 16x + k.$

Also, since the parabola passes through the point (3, 5),

$$5 = 4 \cdot 3^2 - 16 \cdot 3 + k;$$

whence, $k = 17.$

Therefore the equation of the particular parabola that satisfies the three conditions above is

$$y = 4x^2 - 16x + 17.$$

The given relation (1) might have been written in the differential form,

$$d^2y = 8\,dx^2,$$

and y expressed in the form of a multiple integral, namely,

$$y = \int\int 8\,dx^2;$$

whence on integrating,

$$= \int (8x + c)\,dx$$

$$= 4x^2 + cx + k.$$

The former solution is better because it shows all the steps more clearly.

Ex. 2. If s represents distance measured along a straight line, and t time, $\frac{ds}{dt}$ is the velocity of a body that moves in the straight line, and $\frac{d^2s}{dt^2}$ is its acceleration or rate of change of velocity. In the case of a body falling in a vacuum in the neighborhood of the earth's surface, the acceleration or rate of increase in the velocity is constant and equal to about 32.2 feet-per-second per second. The number 32.2 in this connection is denoted by the symbol g. Let it be required to determine s from the known relation,

$$(1)\ \frac{d^2s}{dt^2}=g.$$

This may be written $$\frac{d\left(\frac{ds}{dt}\right)}{dt}=g.$$

Using the differential form, $$d\left(\frac{ds}{dt}\right)=g\,dt,$$

and integrating, $$(2)\ \frac{ds}{dt}=gt+c,$$

in which c is an arbitrary constant of integration.

Writing the latter equation in the differential form,

$$ds=(gt+c)\,dt,$$

and integrating, $$(3)\ s=\tfrac{1}{2}gt^2+ct+k,$$

in which k is another arbitrary constant of integration. In order that the constants c, k may have definite values, two further conditions are required. For instance:

(a) Suppose that the body falls from rest, and that the distance is measured from the starting point.

In this case, $s=0$, and $\frac{ds}{dt}=0$, when $t=0$.

Hence, substituting in (2), $0=0+c$,

that is, $c=0$;

and, substituting in (3), $0=0+0+k$,

that is, $k=0$.

Therefore, the distance through which a body falls in a vacuum on starting from rest is $\frac{1}{2}gt^2$, in which g is about 32.2 and t is the duration of fall in seconds.

(b) Suppose that the body has an initial velocity of 8 feet per second, and that the distance is measured from a point 12 feet above the starting point.

By the last condition, $s=12$ when $t=0$;

and hence, by (3), $12=0+0+k$,

whence $k=12$.

By the other condition, $\frac{ds}{dt} = 8$ when $t = 0$;

and hence, by (2), $8 = 0 + c$, that is, $c = 8$.

Therefore, under these conditions,

$$s = \tfrac{1}{2} gt^2 + 8t + 12.$$

The known relation (1) might have been written in the differential form,

$$d^2s = gdt^2;$$

from this, $$s = \int\int gdt^2;$$

whence, on integration, $$s = \int (gt + c)\, dt$$
$$= \tfrac{1}{2} gt^2 + ct + k.$$

Ex. 3. Evaluate $\int_2^4 \int_1^3 \int_0^2 x^3 (dx)^3$.

The integrations are made *in order from right to left.* Thus, if I denote the integral,

$$\begin{aligned} I &= \int_2^4 \int_1^3 \left[\frac{x^4}{4}\right]_0^2 (dx)^2 = 4\int_2^4 \int_1^3 (dx)^2 \\ &= 4\int_2^4 \Big[x\Big]_1^3 dx = 8\int_2^4 dx \\ &= 16. \end{aligned}$$

Ex. 4. Evaluate $\int_0^2 \int_1^3 \int_2^4 x^3 (dx)^3$. **Ex. 5.** Evaluate $\int_2^4 \int_0^2 \int_1^3 x^3 (dx)^3$.

(Compare Exs. 4, 5, with Ex. 3.)

Ex. 6. Determine all the curves for which $\frac{d^2y}{dx^2} = 0$.

Ex. 7. Find the curve at each of whose points the second derivative of the ordinate with respect to the abscissa is four times the abscissa, and which passes through the origin and the point (2, 4).

Ex. 8. Find $\int_a^b \int_c^d \int_m^n r^4 (dr)^3$. **Ex. 9.** $\int_0^{\frac{\pi}{2}} \int_a^{\beta} \int_0^{\pi} \sin\theta\, (d\theta)^3$.

60. Successive integration with respect to two or more independent variables. In this article the notation commonly used in this kind of integration will be described; and, in preparation for the next article, a few examples will be given so that the student may become familiar with the notation.

Suppose that $$f_1(x, y, z) = \int f(x, y, z)\, dz, \qquad (1)$$

the integration indicated in the second member being performed as if x, y were constants. (It will be remembered that if $x, y, \cdots,$ are independent variables, differentiation of $F(x, y, \cdots,)$ with respect to one of the variables, say x, is performed as if the others were constants.) Then, suppose that

$$f_2(x, y, z) = \int f_1(x, y, z)\, dy, \tag{2}$$

the integration now being performed as if x, z were constants. Again, suppose that

$$f_3(x, y, z) = \int f_2(x, y, z)\, dx, \tag{3}$$

the integration being performed as if y, z were constants. Equation (2), by virtue of equation (1), can be put in the form

$$f_2(x, y, z) = \int \left[\int f(x, y, z)\, dz \right] dy; \tag{4}$$

and equation (3), by virtue of (4), can be written,

$$f_3(x, y, z) = \int \left\{ \int \left[\int f(x, y, z)\, dz \right] dy \right\} dx. \tag{5}$$

The bracketing in the second member of (5) indicates that the differential coefficient, $f(x, y, z)$, is integrated with respect to z; that the result of this integration is then integrated with respect to y; and that, finally, the result of the last integration is integrated with respect to x. In the notation usually adopted, the second member of (5) is abbreviated by removing the brackets, and the order of the variables with respect to which the integrations are made, is indicated by the order of the respective differentials of the variables *beginning at the right and going toward the left.* Thus, the abbreviated form of the second member of (5) is

$$\int\int\int f(x, y, z)\, dx\, dy\, dz. \tag{6}$$

This is a triple integral. Similarly, the double integral in the second member of (4) is generally written,

$$\int\int f(x,\, y,\, z)\, dy\, dz.$$

As to the *integration signs*, the first on the right is taken with the first differential on the right, which is dz in (6) above, the second sign from the right is taken with the second differential from the right, the third sign from the right is taken with the third differential from the right, and so on. It is well to note this usage, because attention must be paid to it when limits of integration are assigned to x, y, z.* In some of the examples below, and often in practical problems, the limits for one variable are functions of one or more of the other variables.

Ex. 1. Evaluate $\int_2^3 \int_1^2 \int_2^5 xy^2\, dx\, dy\, dz.$

If I denote the integral,

$$I = \int_2^3 \int_1 \Big[z \Big]_2^5 xy^2\, dx\, dy = 3 \int_2 \int_1^2 xy^2\, dx\, dy$$

$$= 3 \int_2 \left[\frac{y^3}{3} \right]_1^2 x\, dx = 7 \int_2^3 x\, dx = 17\tfrac{1}{2}.$$

Ex. 2. Evaluate $\int_0^a \int_{2x}^{x^{\frac{3}{2}}} xy^2\, dx\, dy.$

$$I = \int_0^a \left[\frac{y^3}{3} \right]_{2x}^{x^{\frac{3}{2}}} x\, dx = \tfrac{1}{3} \int_0^a (x^{\frac{9}{2}} - 8\, x^3)\, x\, dx$$

$$= \tfrac{2}{3}\, a^5 \left(\tfrac{1}{13}\, a^{\frac{3}{2}} - \tfrac{4}{5}\right).$$

Ex. 3. $\int_2^3 \int_1^2 \int_2^5 xy^2\, dz\, dy\, dx.$ **Ex. 4.** $\int_2^3 \int_1^2 \int_2^5 xy^2\, dz\, dx\, dy.$

(Compare Exs. 3, 4 with **Ex. 1.**)

Ex. 5. $\int_a^{2a} \int_v^{\frac{v^2}{a}} (w + 2\, v)\, dv\, dw.$ **Ex. 7.** $\int_0^b \int_t^{10t} \sqrt{st - t^2}\, dt\, ds.$

Ex. 6. $\int_0^\pi \int_0^{a(1+\cos\theta)} r^2 \sin\theta\, d\theta\, dr.$ **Ex. 8.** $\int_0^{2a} \int_0^x \int_{2y}^{3x} x^2yz\, dx\, dy\, dz.$

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* The notation described above is not universally adopted, but it is the one most frequently used.

61. Application of successive integration to the measurement of areas: rectangular coördinates. In this and the two following articles, problems are solved which show applications of successive integration. In some of the examples there may not be any special advantage in resorting to double integration, for the reason that a single integration may suffice. They are, however, given to the student for the purpose of making him familiar with an instrument for solution which may sometimes be the only one possible. It will be found that the elements in the summations which follow are infinitesimals of a higher order than those which have been met with heretofore.

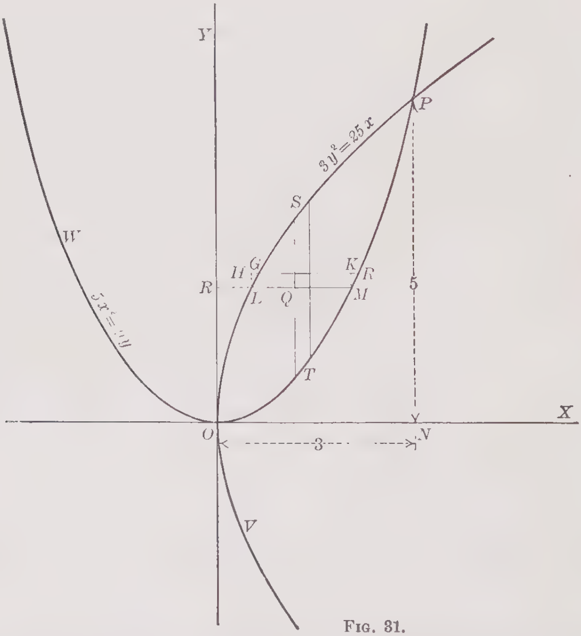

Fig. 31.

Ex. 1. Find the area included between the parabolas whose equations are

$$3y^2 = 25x, \text{ and } 5x^2 = 9y.$$

The parabolas are VOP, WOP. Their points of intersection, O, P, are (0, 0), (3, 5). Taking any point Q within the area as a vertex, construct a rectangle whose sides are parallel to the axes of coördinates and are equal to Δx, Δy. Produce the sides which are parallel to the x-axis until they meet the curves in L, M, G, R, thus forming the strip $LGRM$, and produce ML to meet the y-axis in R. On LM construct the rectangle HM, giving it a width Δy.

$$\text{Area of the rectangle } HM = \text{limit}_{\Delta x \doteq 0} \sum_{x=RL}^{x=RM} \Delta x \, \Delta y.$$

Both y and Δy remain unchanged throughout this summation. Now,

$$RL = \frac{3\,y^2}{25}, \text{ and } RM = 3\sqrt{\frac{y}{5}}.$$

Hence,
$$\text{area } HM = \left[\int_{x=\frac{3y^2}{25}}^{x=3\sqrt{\frac{y}{5}}} dx\right] \Delta y \tag{1}$$

$$= \left(3\sqrt{\frac{y}{5}} - \frac{3\,y^2}{25}\right) \Delta y. \tag{2}$$

As Δy approaches zero, the rectangle HM approaches coincidence with the infinitesimal strip GM, and, in the limit, HM coincides with GM. Also, the area $OLPMO$ is the limit of the sum of all the strips similar to $LGRM$ lying between O and P, when Δy is made to approach zero as a limit. Therefore,

$$\text{area } OLPMO = \text{limit}_{\Delta y \doteq 0} \sum_{y=0}^{y=PN} \left(3\sqrt{\frac{y}{5}} - \frac{3\,y^2}{25}\right) \Delta y$$

$$= \int_{y=0}^{y=5} \left(3\sqrt{\frac{y}{5}} - \frac{3\,y^2}{25}\right) dy \tag{3}$$

$$= 5.$$

If the linear unit is an inch, the answer is in square inches. On substituting for $\left(3\sqrt{\frac{y}{5}} - \frac{3\,y^2}{25}\right)$ in (3) its value as shown by (1) and (2), there is obtained,

$$\text{area } OLPMO = \int_{y=0}^{y=5} \left[\int_{x=\frac{3y^2}{25}}^{x=3\sqrt{\frac{y}{5}}} dx\right] dy$$

$$= \int_0^5 \int_{\frac{3y}{25}}^{3\sqrt{\frac{y}{5}}} dy\, dx. \tag{4}$$

The latter is the customary abbreviated form which indicates that the first integration is made with respect to x between the limits $\frac{3y^2}{25}$ and $3\sqrt{\frac{y}{5}}$ for x, and that the result obtained thereby is to be integrated with respect to y between the limits 0 and 5. The element of area in (4), namely $dy\,dx$, is an infinitesimal of the second order.

Another way of performing the double summation required in adding up all of the elements of area like $dy\,dx$, may be described as follows. Sum all of these elements that are in the vertical strip ST, and then sum all of the vertical strips in $OLPMO$. In the first summation, x and dx do not change, and the upper and lower limits of y are $5\sqrt{\frac{x}{3}}$, $\frac{5x^2}{9}$ respectively; in the second summation, the limits are the values of x at O and P, namely 0 and 3. This double summation is indicated by the double integral

$$\int_0^3\int_{\frac{5x^2}{9}}^{5\sqrt{\frac{x}{3}}} dx\,dy.$$

This, on evaluation, gives an area of 5 square units as before.

The area $OLPMO$ might have been expressed in terms of single integrals. For

$$\begin{aligned} OLPMO &= OLPN - OMPN \\ &= \int_0^3 5\sqrt{\frac{x}{3}}\,dx - \int_0^3 \frac{5x^2}{9}\,dx \\ &= 10 - 5 = 5. \end{aligned}$$

Ex. **2**. Solve Exs. 7–11, Art. 29, by this method.

62. Application of successive integration to the measurement of volumes: rectangular coordinates. If the equation of a surface is given in the form

$$f(x, y, z) = 0,$$

the volume can usually be determined by means of three successive integrations. In the particular case of solids of revolution, the volume can be found by a single integration. This was shown in Art. 30. In Art. 61 the element of area was $dy\,dx$, the area of an infinitesimal rectangle each side of which was an infinitesimal. In the case now to be considered, the element of volume will be the volume, $dx\,dy\,dz$, of an infinitesimal parallelopiped each of whose infinitesimal edges is parallel to one of the axes of coördinates. This will be illustrated in Ex. 1.

Ex. 1. Find the volume of the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let O-ABC be one eighth of the ellipsoid whose volume is required. Then $OA=a$, $OB=b$, $OC=c$. Take IL an infinitesimal distance dx on OX, and through I, L pass the planes HIJ, KLM perpendicular to OX. Take EF of an infinitesimal length dy on ML, and complete the infinitesimal rectangle $EFGD$. Through the lines DE, GF pass planes parallel to the plane ZOX which intersect the curvilinear surface $HJMK$ in the infinitesimal arcs RV, ST. Through a point D' on DR, DD' having an infinitesimal length dz, pass a plane parallel to the plane XOY. The infinitesimal parallelopiped $D'F$, whose volume is $dx\,dy\,dz$, will be taken for the element of volume. The solid O-ACB is the limit of the sum of parallelopipeds of this kind. This limit will now be determined.

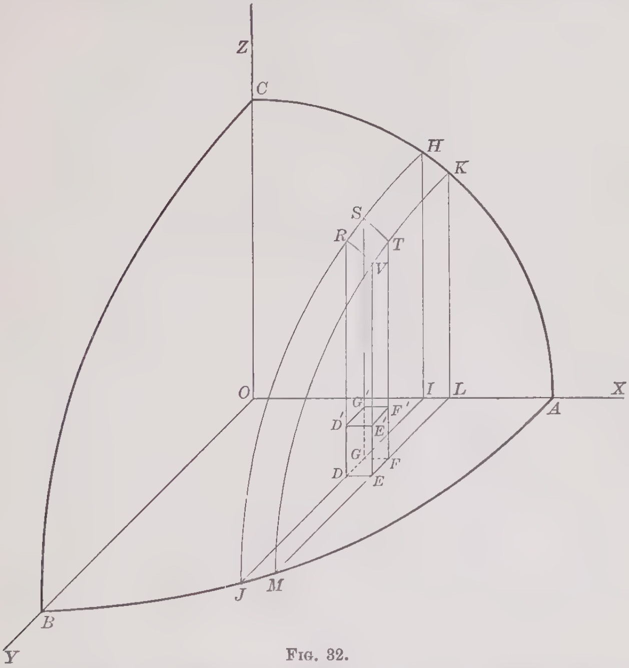

Fig. 32.

First, the volume of the vertical rectangular column RF will be found by adding together all the infinitesimal parallelopipeds such as $D'F$ which are included between $DEFG$ and $RSTV$.

Second, the volume of the slice $HIJMLK$ will be found by adding together all the infinitesimal rectangular columns like RF which are erected between IL and JM.

Third, the volume of $O\text{-}ABC$ will be found by adding together all the infinitesimal slices like $HIJMLK$ that lie between OCB and A.

In the addition of the infinitesimal parallelopipeds from $DEFG$ to $RSTV$, z alone varies, and it varies from zero to DR.

$$\therefore \text{ Vol. } RF = \left[\int\limits_{z=0}^{z=DR} dz\right] dy\,dx. \tag{1}$$

In the addition of the vertical columns from IL to MJ, y alone varies, and it varies from zero to IJ.

$$\therefore \text{ Vol. } HIJMLK = \left\{\int\limits_{y=0}^{y=IJ} \left[\int\limits_{z=0}^{z=DR} dz\right] dy\right\} dx. \tag{2}$$

In the addition of the slices between OCB and A, x varies from zero to OA.

$$\therefore \text{ Vol. } O\text{-}ABC = \int\limits_{x=0}^{x=OA} \left\{\int\limits_{y=0}^{y=IJ} \left[\int\limits_{z=0}^{z=DR} dz\right] dy\right\} dx. \tag{3}$$

Writing this in the usual manner,

$$\text{Vol. } O\text{-}ABC = \int\limits_{x=0}^{x=OA} \int\limits_{y=0}^{y=IJ} \int\limits_{z=0}^{z=DR} dx\,dy\,dz. \tag{4}$$

If the coördinates of R are x, y, z, it follows from the equation of the surface that

$$\begin{aligned} DR &= z, \\ &= c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \end{aligned}$$

At the point J, $z = 0$; and hence,

$$IJ = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Also, $$OA = a.$$

Therefore (4) becomes

$$\text{Vol. } O\text{-}ABC = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz.$$

On making the integrations in their proper order, it is found that

$$\text{Vol. } O\text{-}ABC = \tfrac{1}{6}\pi abc.$$

Hence, the volume of the whole ellipsoid $= \frac{4}{3}\pi abc$.

NOTE 1. The volume of an infinitesimal parallelopiped is an infinitesimal of the third order, the volume of a vertical column is an infinitesimal of the second order, and the volume of a slice is an infinitesimal of the first order.

NOTE 2. Equally well, the planes bounding an infinitesimal slice might have been taken perpendicular to either OZ or OY.

NOTE 3. On putting $a = b = c$, the volume of a sphere of radius a is found to be $\frac{4}{3}\pi a^3$.

Ex. **2**. Find the volume of the ellipsoid given in Ex. 1: (a) by taking the infinitesimal slice at right angles to OY; (b) by taking it at right angles to OZ.

Ex. **3**. Determine the volume of a sphere of radius a by the method of this article.

Ex. **4**. Find the volume bounded by the hyperbolic paraboloid $z = \frac{xy}{c}$, the xy-plane and the planes $x = a$, $x = A$, $y = b$, $y = B$.

Ex. **5**. Find the volume of the wedge cut from the cylinder $x^2 + y^2 = a^2$ by the plane $z = 0$, and the part of the plane $z = x \tan \alpha$ for which z is positive.

Ex. **6**. Find the entire volume bounded by the surface

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1.$$

Ex. **7**. The center of a sphere of radius a is on the surface of a right cylinder the radius of whose base is $\frac{a}{2}$. Find the volume of the part of the cylinder intercepted by the sphere.

63. Further application of successive integration to the measurement of volumes: polar coördinates. The illustration in this article is given, because the use of polar coördinates in dealing with solids is often advantageous. It will be necessary to employ these coördinates in solving some of the problems in Arts. 77, 79.

Ex. **1**. To find the volume of a sphere of radius a by means of polar coördinates. Let a point O on the surface of the sphere be the pole, the tangent plane at O be the xy-plane, and the diameter through O be the z-axis. Take any point P within the sphere, and let its coördinates be denoted by r, θ, ϕ, — r being its distance OP from O, θ the angle that OP makes with the z-axis, and ϕ the angle that the projection of OP on the xy-plane makes with the x-axis. Draw PM perpendicular to OZ. Produce OP an infinitesimal distance dr, and revolve OP through an infinitesimal angle $d\theta$ in the plane ZOP. The point P thus traces an infinitesimal arc of length $rd\theta$. Complete the infinitesimal rectangle PQ that has the sides dr, $rd\theta$ just described.

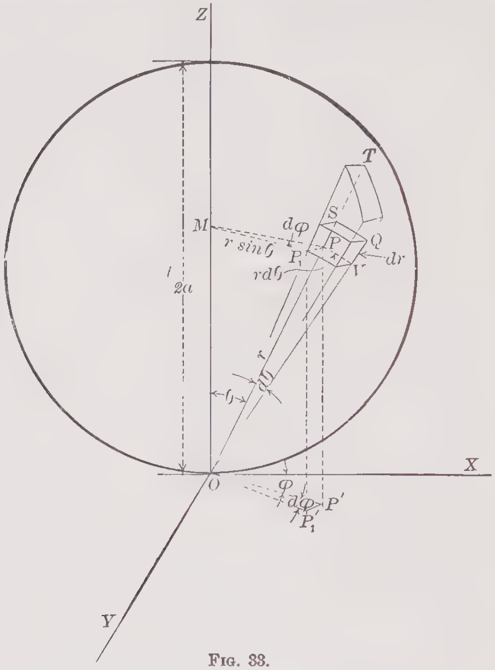

FIG. 33.

Now let the angle ϕ be increased by an infinitesimal amount $d\phi$. Then P and the rectangle move through a distance equal to $MP\,d\phi$, that is $r\sin\theta\,d\phi$. The rectangle will thus have generated a parallelopiped whose edges are dr, $r\,d\theta$, $r\sin\theta\,d\phi$, and whose volume therefore is $r^2\sin\theta\,d\theta\,d\phi\,dr$.* The volume of the sphere is the limit of the sum of such parallelopipeds. Hence,

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* This is not absolutely correct, for the opposite edges of the solid generated differ in length by infinitesimals of higher orders. By fundamental theorems in the differential and integral calculus these differences will not affect the limit of the sum of the infinitesimal solids. See Art. 67 and Note D.

$$\text{Volume of sphere} = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left\{ \int_{\phi=0}^{\phi=2\pi} \left[\int_{r=0}^{r=OT} r^2 dr \right] d\phi \right\} \sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2a\cos\theta} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{8\,a^3}{3} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos^3\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{16\,\pi a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta$$

$$= \tfrac{4}{3}\pi a^3.$$

Ex. **2**. Find the volume of sphere of radius a by this method, on letting the XOY plane pass through the center.

CHAPTER IX

FURTHER GEOMETRICAL APPLICATIONS. MEAN VALUES

64. The calculus has already been employed for the derivation of the equations of curves in Art. 32, for the determination of the areas of curves in Art. 27, for the determination of volumes of solids of revolution in Art. 30, and for the determination of volumes of solids in a more general case, in Arts. 62, 63. Cartesian coördinates were used in all but the last of these applications. This chapter will consider the derivation of the equations of curves and the measurement of areas in cases in which polar coördinates are employed. Special cases in areas and volumes are taken up in Arts. 68–70. The measurement of the lengths of curves for both Cartesian and polar coördinates is considered in Arts. 71, 72; and the measurement of surfaces is discussed in Arts. 74, 75. The subject of mean values is treated in the last two articles of the chapter.

65. Derivation of the equations of curves in polar coördinates. Let the equation of a curve in polar coördinates be

$$f(r, \theta) = 0.$$

It is shown in the differential calculus that, if (r, θ) be any point on the curve, ψ the angle between the radius vector and the tangent at (r, θ), and ϕ the angle that this tangent makes with the initial line, then

$$\tan \psi = r\frac{d\theta}{dr},$$

$$\phi = \psi + \theta,$$

the length of the polar subtangent

$$= r^2 \frac{d\theta}{dr},$$

the length of the polar subnormal

$$= \frac{dr}{d\theta}.$$

Ex. 1. Find the curve in which the polar subnormal is proportional to (is κ times) the sine of the vectorial angle.

In this case, $\frac{dr}{d\theta} = \kappa \sin \theta.$

Using the differential form, $dr = \kappa \sin \theta d\theta,$

and integrating, $r = c - \kappa \cos \theta.$

If $c = \kappa$, $r = \kappa (1 - \cos \theta),$
the equation of the cardioid.

Ex. 2. Find the curve in which the polar subtangent is proportional to (is κ times) the length of the radius vector.

Ex. 3. Find the curve in which the angle between the radius vector and the tangent is n times the vectorial angle. What is the curve when $n = 1$? when $n = \frac{1}{2}$?

66. Areas of curves when polar coördinates are used: by single integration. Let AB be an arc of the curve $r = f(\theta)$, and suppose that angle $AOL = \alpha$, angle $BOL = \beta$. The area of AOB is required.

Divide the angle AOB into n parts, each equal to $\Delta\theta$; then

$$n\Delta\theta = \beta - \alpha.$$

The sector AOB will thus be divided into n sectors, which have equal angles at O. Let POQ be one of these sectors. About O as a center, and with a radius equal to OP, describe through P a circular arc PP_1, which intersects OQ in P_1; and about the same center O with OQ as a radius describe an arc QQ_1, which meets OP in Q_1. The area of the sector POQ is greater than the area of the "interior" sector POP_1, and is less than the area of the "exterior" one QOQ_1. About O as a center, and through each of the points in which the arc AB is intersected by the lines that divide the angle AOB into equal parts, let circular arcs be drawn which intersect the adjacent lines of division on each side, as shown in Fig. 34. There is thus obtained a set of exterior circular sectors like Q_1OQ, and a set of interior circular sectors like POP_1. It has been seen that each sector POQ of the figure AOB is greater than the corresponding interior sector POP_1, and less than the corresponding exterior sector QOQ_1; that is,

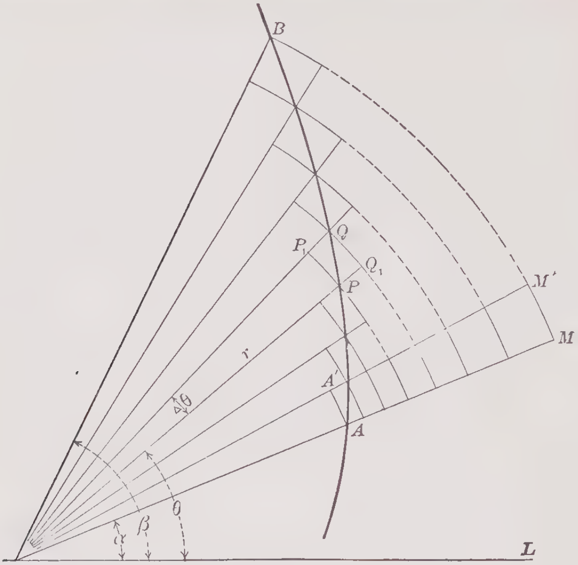

FIG. 34.

$$\text{sector } POP_1 < \text{sector } POQ < \text{sector } Q_1OQ.$$

Therefore the sum of the sectors POQ, which is AOB, is greater than the sum of the interior sectors, and less than the sum of the exterior sectors; that is,

$$\sum_{\theta = \alpha}^{\theta - \beta} POP_1 < AOB < \sum_{\theta = \alpha}^{\theta - \beta} QOQ_1.$$

In the limiting case, when the number of sectors POQ becomes infinite, that is when $\Delta\theta$ approaches zero, the sum of the areas of the interior sectors, and the sum of the areas of the exterior sectors approach equality. For, the difference between these two sums is equal to the area of $AMM'A'$, which is $\frac{1}{2}(OM^2 - OA^2)\Delta\theta$, and can therefore be made as small as one pleases by decreasing $\Delta\theta$. The area AOB always lies between these sums; and hence,

$$\text{area } AOB = \text{limit}_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} POP_1.$$

If the coördinates of P be denoted by r, θ, the area of

$$POP_1 = \tfrac{1}{2} r^2 \Delta\theta.$$

Hence, $$\text{area } AOB = \text{limit}_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2} r^2 \Delta\theta;$$

that is, $$\textbf{area } \boldsymbol{AOB} = \frac{1}{2}\int_\alpha^\beta r^2 d\theta,$$

by the definition of a definite integral. The element of area for polar coördinates is thus $\frac{1}{2} r^2 d\theta$.

Ex. 1. Find the area of the sector of the logarithmic spiral whose equation is $r = e^{a\theta}$, between the radii vectores for which $\theta = \alpha$, $\theta = \beta$.

$$\text{Area } POQ = \tfrac{1}{2}\int_\alpha^\beta r^2 d\theta$$

$$= \tfrac{1}{2}\int_\alpha^\beta e^{2a\theta} d\theta$$

$$= \frac{e^{2a\beta} - e^{2a\alpha}}{4a}$$

$$= \frac{r_2{}^2 - r_1{}^2}{4a},$$

r_1, r_2 being the bounding radii vectores.

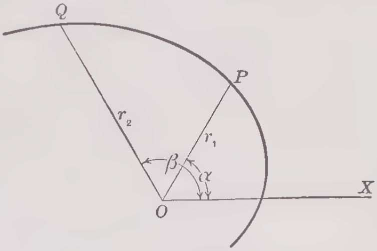

FIG. 35.

Ex. 2. Find the area of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

The area of one half the loop,

$$OMA = \tfrac{1}{2}\int r^2 d\theta,$$

between proper limits for θ, which must be determined.

The initial and final positions of the radius vector are OA and OL the tangent to the arc OM at O. For $r = 0$, the equation of the curve gives

$$0 = a^2 \cos 2\theta;$$

and hence, $$2\theta = \pm\frac{\pi}{2}, \text{ or } \theta = \pm\frac{\pi}{4}.$$

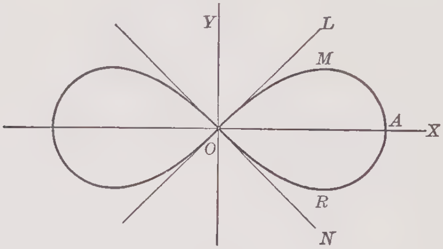

FIG. 36.

The positive sign indicates the position of OL, and the negative sign that of ON.

Hence, $$\text{area } OMA = \tfrac{1}{2}\int_0^{\frac{\pi}{4}} r^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= \frac{a^2}{4}.$$

Hence, $$\text{area of loop } OMARO = \frac{a^2}{2}.$$

Ex. **3.** Find the area of a sector of the spiral of Archimedes, $r = a\theta$, between $\theta = \alpha$, $\theta = \beta$.

Ex. **4.** Find the area of the part of the parabola $r = a \sec^2\frac{\theta}{2}$ intercepted between the curve and the latus rectum.

Ex. **5.** Find the area of the cardioid $r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}$.

Ex. **6.** Find the area of the loop of the folium of Descartes,

$$x^3 + y^3 - 3axy = 0.$$

(Hint: Change to polar coördinates, thus obtaining $r = \dfrac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$; and then change the variable θ by putting $z = \tan\theta$.)

Ex. **7.** Show that the area bounded by any two radii vectores of the hyperbolic spiral $r\theta = a$, is proportional to the difference between the lengths of these radii.

Ex. **8.** Show that the area of a loop of the curve $r^2 = a^2\cos n\theta$ is $\dfrac{a^2}{n}$.

67. Areas of curves when polar coördinates are used: by double integration. The areas of curves whose equations are given in polar coördinates can be found by double integration, in a manner analogous to that used in Art. 61. Example 1 below will serve to make the method plain. Successive integration in two variables, polar coördinates, will also be required in Arts. 77, 79.

Ex. 1. Find by double integration the area of the circle whose equation is $r = 2a\cos\theta$.

Let $OLAN$ be the given circle, O being the pole and OA the diameter $2a$. Within the circle take any point P with coördinates (r, θ). Draw OP and produce it a distance Δr to S. Revolve the line OPS about O through an angle $\Delta\theta$ to the position OQR. Then

$$\text{area } PQRS = \tfrac{1}{2}[(r+\Delta r)^2 - r^2]\,\Delta\theta$$
$$= r\,\Delta r\,\Delta\theta + \tfrac{1}{2}(\Delta r)^2\,\Delta\theta.$$

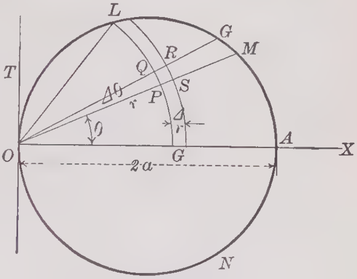

Fig. 37.

Produce OP, OQ to meet the circle in M, G. The area of the sector MOG will be found by adding all the elements $PQRS$ therein, and the area of the semicircle $OLMA$ will be calculated by adding all the sectors like MOG that it contains.

$$\text{Area } MOG = \Sigma PQRS$$

$$= \text{limit}_{\Delta r \doteq 0} \sum_{r=0}^{r=OM} \left(r\,\Delta r\,\Delta\theta + \tfrac{1}{2}(\Delta r)^2\,\Delta\theta\right)$$

$$= \text{limit}_{\Delta r \doteq 0} \sum_{r=0}^{r=OM} r\,\Delta r\,\Delta\theta$$

by a fundamental theorem in the calculus,*

and hence, $\quad \text{area } MOG = \displaystyle\int_{r=0}^{r=OM} r\,dr\,\Delta\theta,$

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* See Note D, Appendix.

in which the integration is performed with respect to r. Hence,

$$\text{area } OLMA = \int_{\theta=0}^{\theta=TOX}\left[\int_{r=0}^{r=OM} r\,dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\,d\theta\,dr$$

$$= \frac{\pi a^2}{2}.$$

Hence, the area of the circle is πa^2.

The area of OMA can also be obtained by finding the area of the circular strip LG whose arcs are distant r, $r + dr$ from O, and then adding all of the similar concentric circular strips from O to A. The angle $LOG = \cos^{-1}\frac{r}{2a}$. It will be found that

$$\text{area } OMA = \int_0^{2a}\int_0^{\cos^{-1}\left(\frac{r}{2a}\right)} r\,dr\,d\theta = \frac{\pi a^2}{2} \text{ as before.}$$

Ex. 2. Find by double integration the area of the circle of radius a, the pole being at the center; (1) by adding equiangular sectors; (2) by adding concentric circular strips.

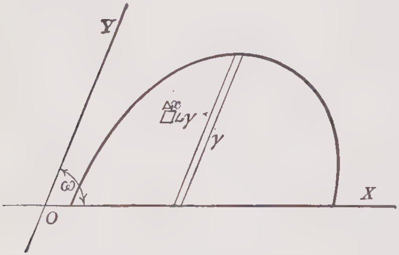

Fig. 38.

68. Areas in Cartesian coördinates with oblique axes. In this case the method of finding the area is similar to that in Art. 61. Let the axes be inclined at an angle ω, and construct a parallelogram whose sides are parallel to the axes and have the lengths Δx, Δy. The area of this parallelogram is

$$\Delta x\,\Delta y \sin\omega.$$

The whole area is the limit of the sum of all parallelograms that are constructed within the perimeter of the figure when Δx and Δy approach zero. Hence, the area is the value of the double integral

$$\int\int \sin\omega\,dx\,dy, \quad \text{that is,} \quad \sin\omega\int\int dx\,dy,$$

between the proper limits for x and y. If the element of area be an infinitesimal strip parallel to the y-axis, as in Art. 27, the area is the value of the integral

$$\sin\omega\int y\,dx$$

between the proper limits for x.

69. Integration after a change of variable. Integration when the variables are expressed in terms of another variable. The function under the sign of integration may assume a much simpler form on changing the variables. Examples 1, 3 illustrate this. Sometimes, also, the ordinary variables are expressed in terms of another variable. Examples 2, 4 illustrate this. When a change is made in the variable or variables, the corresponding changes should be made in the limits. The work of returning to the original variables, in order to substitute the original limits, will thus be avoided. These remarks are also applicable to practical examples in other articles.

Ex. 1. Determine the area of the circle whose equation is $x^2 + y^2 = a^2$. (Compare Ex. 3, Art. 27.)

$$\text{Area of circle} = 4\int_0^a y\,dx.$$

Put $x = a\cos\theta$. Then $y = a\sin\theta$, and $dx = -a\sin\theta\,d\theta$.

Also, $\theta = \frac{\pi}{2}$ when $x = 0$, and $\theta = 0$ when $x = a$.

Making these substitutions in the integral above,

$$\text{area of circle} = -4\int_{\frac{\pi}{2}}^{0} a^2\sin^2\theta\,d\theta$$

$$= -4a^2\int_{\frac{\pi}{2}}^{0}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \pi a^2.$$

Ex. 2. Find the area between the x-axis and the complete arch of the cycloid whose equations are $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

$$\text{Area} = \int_0^{2\pi a} y\,dx.$$

When $x = 0,\ \theta = 0$; and when $x = 2\pi a,\ \theta = 2\pi$.

Also, $dx = a(1 - \cos\theta)\,d\theta$.

If these values for y, dx, and the limits, be substituted in the integral above, it becomes

$$\begin{aligned}\text{area} &= a^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta\\ &= a^2\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta\\ &= 3\pi a^2.\end{aligned}$$

That is, the area is three times that of the generating circle.

Ex. 3. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Compare Ex. 3, Art. 27.)

(Hint: put $x = a\cos\phi$, then $y = b\sin\phi$.)

Ex. 4. Find the volume of the solid generated by the revolution of a complete arch of the cycloid of Ex. 2 about the x-axis.

70. Measurement of the volumes of solids by means of infinitely thin cross-sections. In Art. 30 the volume of a solid of revolution was determined by finding the volume of an infinitely thin slice of the solid, the slice being taken at right angles to the axis of the figure, and the sum of the volumes of all such slices being then found. This method can be extended to other figures besides figures of revolution. Some convenient line is chosen, and an infinitesimally thin slice of the solid is taken at right angles to this line. If the area of a face of the thin slice can be expressed in terms of its distance from some point on the line, the volume of the slice can be expressed in terms of this distance; and from this, the sum of the volumes of all the slices can be found. For example, let the chosen line be taken for the axis of x, and suppose that the area of a face of a thin slice at right angles to this line is $f(x)$. Let the thickness of the slice be Δx. The volume is (as in Art. 30) the limit of the sum of an infinite number of infinitesimal cylinders whose volumes are of the form $f(x)\,\Delta x$. That is,

$$\text{volume of the solid} = \text{limit}_{\Delta x \doteq 0} \Sigma f(x)\,\Delta x,$$

$$= \int f(x)\,dx,$$

in which the limits of integration are determined from the figure.

Ex. **1.** Determine by this method the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(The student is advised to make a figure.) At a distance x from the center cut out a very thin slice at right angles to the x-axis, and let its thickness be dx. The face of this cylindrical slice will be an ellipse whose semiaxes are

$$c\sqrt{1 - \frac{x^2}{a^2}}, \quad b\sqrt{1 - \frac{x^2}{a^2}}.$$

These values are deduced from the equation of the ellipsoid.

$$\text{The area of this ellipse} = \pi bc\left(1 - \frac{x^2}{a^2}\right).$$

Hence, $$\text{the volume of the slice} = \pi bc\left(1 - \frac{x^2}{a^2}\right)\Delta x;$$

and therefore, $$\text{volume of ellipsoid} = \pi bc \int_{-a}^{+a}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= \tfrac{4}{3}\pi abc.$$

Ex. **2.** Find the volume of a sphere of radius a by this method.

Ex. **3.** Find the volume of the torus generated by revolving about the x-axis the circle $x^2 + (y - b)^2 = a^2$, in which $b > a$.

Ex. **4.** Find the volume of a pyramid or a cone having a base B and a height h.

Ex. **5.** Find the volume of a right conoid with a circular base and altitude h, the radius of the base being a.

Ex. **6.** A rectangle moves from a fixed point, one side varying as the distance from this point, and the other as the square of this distance. At the distance of 2 feet, the rectangle becomes a square of 3 feet. What is the volume then generated?

Ex. **7.** Given a right cylinder of altitude h, and radius of base a. Through a diameter of the upper base two planes are passed touching the lower base on opposite sides. Find the volume included between the planes.

Ex. **8.** Find the volume of the elliptic paraboloid $2x = \frac{y^2}{p} + \frac{z^2}{q}$ cut off by the plane $x = h$.

71. Lengths of curves: rectangular coördinates. To find the length of a curve is equivalent to finding the straight line that has the same length as the curve. For this reason the measurement of its length is usually called "the rectification of the curve."* The deduction here made of the integration formulæ for the length of a curve depends upon the definition that integration is a process of summation.

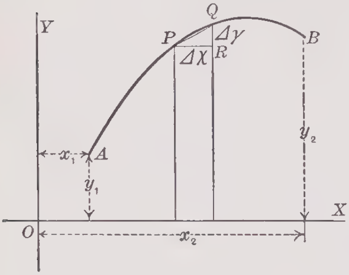

FIG. 39.

The equation of a given curve is $f(x, y) = 0$; it is required to find the length s of an arc AB, A being the point (x_1, y_1), B being the point (x_2, y_2). On the curve take any two points P, Q whose coördinates are x, y, and $x + \Delta x$, $y + \Delta y$. Draw the chord PQ and make the construction indicated in the figure.

$$\text{The chord } PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{1}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x. \tag{2}$$

As Q approaches infinitely near to P, that is, when Δx approaches zero, the chord PQ approaches coincidence with the arc PQ. It is shown in the differential calculus that if Δx is an infinitesimal of the first order, the difference between the arc and its chord is an infinitesimal of at least the third order; that is,

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* In 1659 Wallis (see footnote, Art. 27) published a tract in which he showed a method by which curves could be rectified, and in 1660 one of his pupils, William Neil, found the length of an arc of the semi-cubical parabola $x^3 = ay^2$. This is the first curve that was rectified. Before this it had been generally supposed that no curve could be measured by a mathematical process. The second curve whose length was found is the cycloid. Its rectification was effected by Sir Christopher Wren (1632–1723) and published in 1673. This was before the development of the calculus by Leibniz and Newton.

$$\text{arc } PQ = \text{chord } PQ + i_3, \tag{3}$$

in which i_3 is an infinitesimal of the third order when Δx is an infinitesimal of the first order.

Therefore, $s = \Sigma(\text{arcs } PQ)$

$$= \text{limit}_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} \left(\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x + i_3 \right)$$

$$= \text{limit}_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x,$$

by a fundamental theorem.* As Δx approaches zero, $\frac{\Delta y}{\Delta x}$ in general approaches a definite limiting value, namely, $\frac{dy}{dx}$. Therefore, by the definition of a definite integral,

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{4}$$

In applying this formula it will be necessary to express $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ in terms of x before integration is attempted.

Instead of being put in the form (2), equation (1) may be given the form,

$$\text{chord } PQ = \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2} \Delta y.$$

By the same reasoning as above, it can then be shown that

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \tag{5}$$

in which $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$ must be expressed in terms of y before integration is performed. Formula (4) or formula (5) will be used, according as it is more convenient to take x or y for the independent variable.

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* See Note D, Appendix.

If Δs denotes the length of the arc PQ, it follows from (3) that

$$\frac{\Delta s}{\Delta x}=\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}+\frac{i_3}{\Delta x}.$$

Therefore, by the differential calculus,

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

whence, $$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Similarly, $$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

In order to recall formulæ (4) and (5) immediately whenever they may happen to be required, the student need only remember the construction of the triangle PQR, and let its sides become infinitesimal.

Ex. 1. Find the length of the circle whose equation is $x^2 + y^2 = a^2$. Let AB be the first quadrantal arc of the circle.

In this case, $$\frac{dy}{dx}=-\frac{x}{y}.$$

Hence, $$\text{arc } AB=\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^a\sqrt{1+\frac{x^2}{y^2}}\,dx$$

$$=\int_0^a\sqrt{\frac{x^2+y^2}{y^2}}\,dx=a\int_0^a\frac{dx}{\sqrt{a^2-x^2}}.$$

$$=a\left[\sin^{-1}\frac{x}{a}\right]_0^a=\frac{\pi a}{2}.$$

Therefore, the perimeter of the circle $(=4\,AB)=2\,\pi a$.

Ex. 2. Find the length of the arc of the parabola from the vertex to the point (x_1, y_1). Find the length of the arc from the vertex to the end of the latus rectum.

Ex. 3. Find the length of the arc of the semicubical parabola $ay^2 = x^3$ from the origin to the point (x_1, y_1). Also to the point for which $x = 5a$.

Ex. 4. Find the length of the arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from the vertex to the point (x_1, y_1). Also to the point for which $x = a$.

Ex. 5. Find the length of the arc of the cycloid from the point at which $\theta = \theta_0$ to the point at which $\theta = \theta_1$. Also find the length of a complete arch of the curve.

Ex. 6. Find the entire length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 7. Show that in the ellipse

$$x = a \sin\phi, \quad y = b \cos\phi,$$

ϕ being the complement of the eccentric angle of the point (x, y), the arc s measured from the extremity of the minor axis is

$$s = a\int_0^{\phi} \sqrt{1 - e^2 \sin^2\phi}\, d\phi,$$

and that the entire length of the ellipse is

$$4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2\phi}\, d\phi,$$

in which e is the eccentricity.*

72. Lengths of curves: polar coördinates. The equation of a curve is $f(r, \theta) = 0$, and the length s of the arc AB is required, A being the point (r_1, θ_1), and B the point (r_2, θ_2). On the curve take any two points P, Q, whose coördinates are r, θ, $r + \Delta r$, $\theta + \Delta\theta$. Draw OP, OQ, and the chord PQ. About O as a center, and with a radius equal to OP, describe the arc PR which intersects OQ in R, and draw PR_1 at right angles to OQ. Then the angle

$$POQ = \Delta\theta, \quad RQ = \Delta r,$$

and arc

$$PR = r\Delta\theta.$$

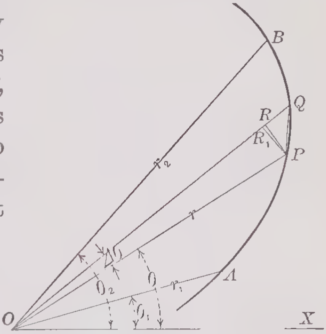

FIG. 40.

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* This integral, which is known as "the elliptic integral of the second kind," cannot be expressed, in a finite form, in terms of the ordinary functions of mathematics. See Ex. 8, Art. 83.

It is shown in the differential calculus that when $\Delta\theta$ is an infinitesimal of the first order,

$$PR_1 = \text{arc } PR - i_3, \text{ an infinitesimal of the third order;}$$

$$QR_1 = QR + i_2, \text{ an infinitesimal of the second order;}$$

$$\text{chord } PQ = \text{arc } PQ - i'_3, \text{ an infinitesimal of the third order.}$$

In the right-angled triangle PR_1Q,

$$\text{chord } PQ = \sqrt{PR_1^2 + R_1Q^2}.$$

Hence, when $\Delta\theta$ approaches zero,

$$\text{arc } PQ - i'_3 = \sqrt{(PR - i_3)^2 + (RQ + i_2)^2},$$

$$\text{or, } \quad \text{arc } PQ = \sqrt{(r\Delta\theta - i_3)^2 + (\Delta r + i_2)^2} + i'_3 \qquad (1)$$

$$= \sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2 - \frac{2\,ri_3}{\Delta\theta} + 2\,i_2\frac{\Delta r}{(\Delta\theta)^2} + \frac{i_3^{\,2} + i_2^{\,2}}{(\Delta\theta)^2}}\;\Delta\theta + i'_3, \qquad (2)$$

which differs by an infinitesimal of at least the first order from

$$\sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2}\;\Delta\theta.$$

Therefore,

$$s = \Sigma PQ = \text{limit}_{\Delta\theta \doteq 0}\sum_{\theta=\theta_1}^{\theta=\theta_2}\sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2}\cdot\Delta\theta,$$

when $\Delta\theta$ approaches zero, $\frac{\Delta r}{\Delta\theta}$ approaches the definite limiting value $\frac{dr}{d\theta}$. Therefore, by the definition of a definite integral,

$$\boldsymbol{s = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.} \qquad (3)$$

It will be necessary to express $\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ in terms of θ before integration is made.

Similarly, on removing Δr from the radical sign in (1), it can be shown that

$$\boldsymbol{s = \int_{r_1}^{r_2}\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr,} \qquad (4)$$

in which $\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$ must be expressed in terms of r before integration is made. Formula (**3**) or (**4**) will be used according as it is more convenient to take θ or r for the independent variable.

In order to recall these formulæ immediately it is only necessary for the student to remember the construction of the figure PRQ, and to suppose that its sides are infinitesimal.

Ex. **1.** Find the length of the circumference of the circle whose equation is

$$r = a.$$

Here $$\frac{dr}{d\theta} = 0.$$

Hence, $$s = \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= a\int_0^{2\pi} d\theta$$
$$= 2\,\pi a.$$

Ex. **2.** Find the length of the circle of which the equation is $r = 2\,a\sin\theta$.

Ex. **3.** Find the entire length of the cardioid, $r = a(1-\cos\theta)$.

Ex. **4.** Find the arc of the spiral of Archimedes, $r = a\theta$, between the points (r_1, θ_1), (r_2, θ_2).

Ex. **5.** Find the length of the hyperbolic spiral, $r\theta = a$, from (r_1, θ_1) to (r_2, θ_2).

Ex. **6.** Find the length of the logarithmic spiral, $r = e^{a\theta}$, from $(1, 0^c)$ to (r_1, θ_1).

Ex. **7.** Find the length of the arc of the cissoid $r = 2\,a\tan\theta\sin\theta$ from the cusp $(\theta = 0)$ to $\theta = \frac{\pi}{4}$.

Ex. **8.** Find the length of the arc of the parabola $r = a\sec^2\frac{\theta}{2}$ from $\theta = 0$ to $\theta = \theta_1$; also, from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

73. The intrinsic equation of a curve. Let PQ be the arc of a given curve, and let s denote its length. Suppose a point starts at P and moves along the curve towards Q. At the instant of starting the point moves in the direction of the tangent PT_1. In passing over the arc PQ the direction of motion changes at every instant, until at Q the point is moving in the direction of the tangent QT_2. The total change in direction, as the point moves from P to Q, is measured by the angle ϕ between the two tangents. It will be found that a relation exists between the distance s through which the point has moved, and the angle ϕ by which the direction of its motion has changed. This relation between s and ϕ is called the *intrinsic equation of the curve.* The form of this equation depends only on the nature of the curve, and the choice of the initial point P. On the other hand, the form of the equation of a curve in other systems of coördinates, for example the rectangular and polar, depends upon points and lines that are independent of the curve. Hence the term "intrinsic."

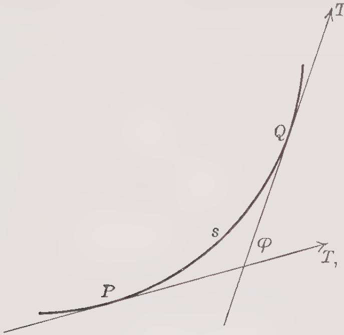

Fig. 41.

To find the intrinsic equation of a curve given in rectangular or polar coördinates,

(1) Determine the length of arc s measured from some convenient starting point up to a variable point on the curve.

(2) Find the angle ϕ between the tangents at the initial and the terminal points.

(3) Eliminate the rectangular or polar variables from the equations thus found.

Ex. 1. Find the intrinsic equation of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

If the vertex of the curve be taken as starting point,

$$(1)\quad s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right). \quad \text{[Ex. 4, Art. 71.]}$$

Also, since the tangent at the vertex is parallel with the x-axis,

$$(2)\quad \tan\phi = \frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

The elimination of x from (1) and (2) gives the required equation between s and ϕ, viz.,

$$s = a\tan\phi.$$

It is easy to extend this result and show that

$$s = a\left[\tan(\phi + \phi_1) - \tan\phi_1\right]$$

is the intrinsic equation of the catenary when any point A is chosen for the initial point. The angle ϕ_1 is then the angle between the tangent at the vertex and the tangent at the point A.

Ex. **2.** Find the intrinsic equation of the parabola

$$r = a\sec^2\frac{\theta}{2}.$$

If the vertex of the parabola be taken for the initial point,

$$(1)\quad s = a\tan\frac{\theta}{2}\sec\frac{\theta}{2} + a\log\tan\left(\frac{\theta}{4} + \frac{\pi}{4}\right). \quad \text{[Ex. 8, Art. 72.]}$$

Also, since the tangent at the vertex makes an angle of $\frac{\pi}{2}$ with the polar axis,

$$\phi = \phi' - \frac{\pi}{2},$$

where ϕ' is the angle that the tangent at the point (r, θ) makes with the polar axis. But

$$2\phi' = \theta + \pi.$$

Hence,
$$\theta = 2\phi.$$

On substituting this value of θ in equation (1), the intrinsic equation of the parabola is found to be

$$s = a\tan\phi\sec\phi + a\log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right).$$

EXAMPLES.

1. Find the intrinsic equation of a circle with radius r.

2. Find the intrinsic equation of the cardioid $r = a(1 - \cos\theta)$, the arc being measured from the polar origin.

3. Find the intrinsic equation of the cycloid

$$\left.\begin{aligned} x &= a(\theta - \sin\theta), \\ y &= a(1 - \cos\theta), \end{aligned}\right\}$$

(1) the origin being the initial point,
(2) the vertex being the initial point.

4. Find the intrinsic equation of the parabola $y^2 = 4px$,
(1) the vertex being the initial point,
(2) the extremity of the latus rectum being the initial point.

5. Find the intrinsic equation of the semicubical parabola $3ay^2 = 2x^3$, taking the origin for initial point.

6. Find the intrinsic equation of the curve $y = a \log \sec \frac{x}{a}$, taking the origin for the initial point.

7. Find the intrinsic equation of the logarithmic spiral $r = ae^{c\theta}$.

8. Find the intrinsic equation of the tractrix

$$x = \sqrt{c^2 - y^2} + c \log \frac{c + \sqrt{c^2 - y^2}}{y},$$

taking the point $(0, c)$ as the initial point.

9. Find the intrinsic equation of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, taking any one of the cusps as initial point.

74. Areas of surfaces of solids of revolution. Suppose that the surface is generated by the revolution about the x-axis of the arc AB of the curve whose equation is $y = f(x)$; and let the coördinates of the points A, B, be x_1, y_1, and x_2, y_2, respectively. Take (Fig. 42) any two points on the curve, say P, Q, whose coördinates are x, y, and $x + \Delta x$, $y + \Delta y$. Draw the chord PQ

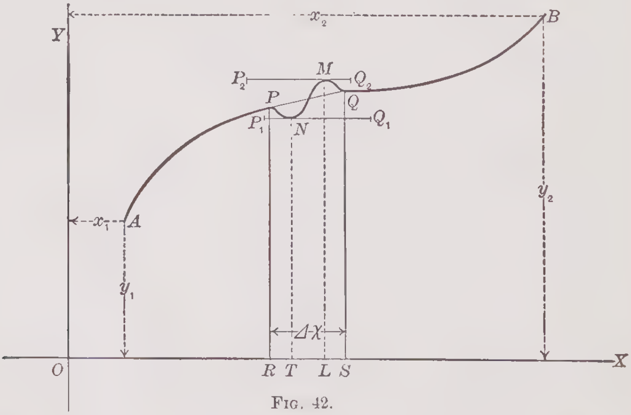

Fig. 42.

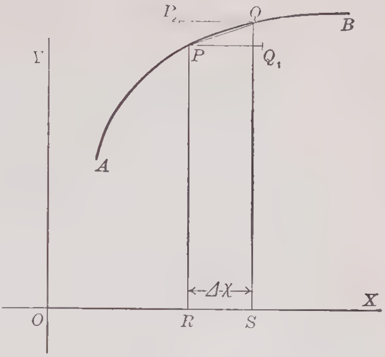

FIG. 43.

and the ordinates RP, SQ, and suppose that LM is an ordinate which is not less, and that TN is an ordinate which is not greater than any ordinate that can be drawn from the arc $PNMQ$ to the x-axis. (In Fig. 43, LM coincides with SQ, and TN coincides with RP.) Through N, M, draw lines P_1Q_1, P_2Q_2, parallel to the x-axis and equal in length to the arc $PNMQ$. On the revolution of the arc AB about OX, each point in AB describes a circle with its ordinate as radius.

The surface generated by arc $PNMQ$

$$\leqq 2\pi LM \times \text{arc } PQ,$$

and

$$\geqq 2\pi TN \times \text{arc } PQ.$$

When Δx is an infinitesimal of the first order,

arc $PQ =$ chord $PQ + i_3$, an infinitesimal of at least the third order;

$LM = RP + i$, an infinitesimal of at least the first order;

$TN = RP - i'$, an infinitesimal of at least the first order.

Hence, since the chord $PQ = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$, it follows that

$$2\pi\left(y - i'\right)\left(\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x + i_3\right)$$

$$\leqq \text{surface generated by arc } PQ$$

$$\leqq 2\pi\left(y + i\right)\left(\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x + i_3\right).$$

Therefore,

$$\text{limit}_{\Delta x \doteq 0}\sum_{x=x_1}^{x=x_2} 2\pi\left(y - i'\right)\left(\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x + i_3\right)$$

$$\overline{\overline{<}} \text{ surface generated by } AB$$

$$\overline{\overline{<}} \operatorname{limit}_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} 2\pi\left(y+i\right)\left(\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\cdot \Delta x + i_3\right).$$

By a fundamental theorem * the least and the greatest expressions in this inequality are each equal to

$$\operatorname{limit}_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} 2\pi y\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\cdot \Delta x.$$

Hence, surface generated by AB

$$= \operatorname{limit}_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} 2\pi y\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x.$$

When Δx approaches zero, $\frac{\Delta y}{\Delta x}$ takes a definite limiting value, namely $\frac{dy}{dx}$. Therefore, by the definition of a definite integral,

$$\textbf{area of surface} = \int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx. \qquad (1)$$

It is necessary to express the function under the sign of integration in terms of x before integration is performed. If

$$\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}\,\Delta y$$

be used for the length of the chord, there will result,

$$\textbf{area of surface} = \int_{y_1}^{y_2} 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy. \qquad (2)$$

Formula (1) or formula (2) is taken according as it is more convenient to choose x or y for the independent variable.

The surface generated by the revolution of AB about the y-axis is given by the formulæ,

---

* See Note D, Appendix.

$$\text{surface} = \int_{y_1}^{y_2} 2\,\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy, \tag{3}$$

$$\text{surface} = \int_{x_1}^{x_2} 2\,\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{4}$$

The student is advised to deduce these formulæ for himself. The expressions under the sign of integration in formulæ (1), (2) may both be written $2\,\pi y\, ds$ by Art. 71, and those in formulæ (3), (4) may both be written $2\,\pi x\, ds$. In order to recall immediately a formula for the area of a surface of revolution, it is only necessary to remember that the area traced out by an infinitesimal arc in its revolution about any line is equal to the product of the length of the infinitesimal arc by the length of the circle which is described by a point on the arc.

Ex. 1. Find the surface generated by the revolution of a semicircle of radius a about its diameter.

Let the diameter be the x-axis, and the origin be at the center; the equation of the curve will be

$$x^2 + y^2 = a^2.$$

Surface generated by ABA' about x-axis

$$= 2\,\pi \int_{-a}^{+a} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

But, $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2}$

$$= \frac{x^2 + y^2}{y^2} = \frac{a^2}{y^2}.$$

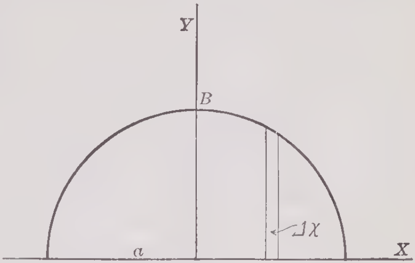

FIG. 44.

Hence, $\text{surface} = 2\,\pi a \int_{a}^{+a} dx$

$$= 4\,\pi a^2.$$

Ex. 2. Find the surface of the prolate spheroid obtained by revolving about the x-axis the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Ex. 3. Find the surface generated by revolving about the x-axis the parabola $y^2 = 4\,ax$. Show that the curved surface of the figure generated by the arc between the vertex and the latus rectum is 1.219 times the area of its base.

Ex. 4. Find the surface generated by revolving about the y-axis the catenary $$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \text{ from } x = 0 \text{ to } x = a.$$

Ex. 5. Find the entire surface generated by revolving about the x-axis the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 6. A quadrant of a circle of radius a revolves about the tangent at one extremity. Find the area of the curved surface generated.

Ex. 7. The cardioid $r = a(1 + \cos\theta)$ revolves about the initial line. Find the area of the surface generated.

75. Areas of surfaces whose equations have the form $z = f(x, y)$. Areas of surfaces of revolution were considered in the last article. A more general case will now be discussed. In the explanation of the following method for measuring the area of a surface, reference will be made to these two geometrical propositions:

(a) The area of the orthogonal projection of a plane area upon a second plane is equal to the area of the portion projected multiplied by the cosine of the angle between the planes. (See C. Smith, *Solid Geometry*, Art. 31.)

(b) If the equation of a surface be in the form $z = f(x, y)$, the cosine of the angle between the xy-plane and the tangent plane at any point (x, y, z) of the surface is

$$\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{-\frac{1}{2}}.$$

(See C. Smith, *Solid Geometry*, Arts. 206, 26.)

Let $z = f(x, y)$ be the equation of the surface $BFCMALB$ whose area is required. Take two points P, Q, whose coördinates are x, y, z, $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, respectively. Through P and Q pass planes parallel to the yz-plane and let them intersect the surface in the arcs ML, M_1L_1. Also pass planes through P, Q, parallel to the zx-plane. The curvilinear figure PQ is thus formed. The projection of the surface PQ on the xy-plane is the rectangle P_1Q_1 whose area is $\Delta x \, \Delta y$. When Δx, Δy approach zero, the point Q comes infinitely close to P; and the curvilinear surface PQ, which is then infinitesimal, *approaches coincidence with that portion of the tangent plane at* P, which also has P_1Q_1 for its projection on the xy-plane. The area of P_1Q_1 also becomes $dx\,dy$.

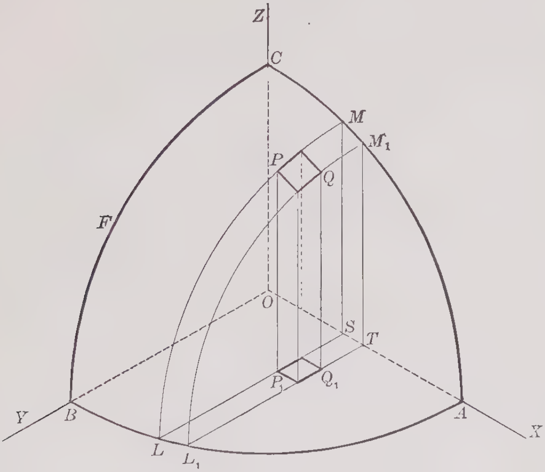

FIG. 45.

Now let Q be infinitely near to P. If γ is the angle between the xy-plane and the tangent plane at P, it follows from (a) and the remarks which have just been made, that

$$\text{area } P_1Q_1 = \text{area } PQ \cdot \cos\gamma.$$

Hence,
$$\text{area } PQ = \text{area } P_1Q_1 \cdot \sec\gamma$$
$$= dx\,dy\sec\gamma.$$

Therefore, by (b), $\text{area } PQ = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$

The summation of all the infinitesimal surfaces PQ in the strip LMM_1L_1 gives

$$\text{area of strip } LM_1 = \left[\int_{y=0}^{y=SL}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dy\right]\Delta x.$$

The summation of all the strips like LM_1 in the surface $BFCMALB$ gives

$$\text{area of surface } BFCMALB = \int_{x=0}^{x=OA} \left[\int_{y=0}^{y=SL} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dy \right] dx;$$

or, abbreviating in the usual way,

$$= \int_0^{OA} \int_0^{SL} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx\, dy.$$

The limits $y = SL$, $x = OA$ can be determined from the equation of the surface. It is necessary to express the function under the signs of integration in terms of x and y. It may happen that a more convenient form of the equation of the surface is either $x = f(y, z)$, or $y = f(z, x)$. The area of the surface will then be the value of either one or the other of the double integrals

$$\int\int \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} dy\, dz, \quad \int\int \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dz\, dx,$$

between the proper limits of integration.

In some cases, there are two surfaces each of which intercepts a portion of the other. In finding the area of the intercepted portion of one of the surfaces, it is necessary to obtain the partial derivatives that are required in the formulæ of integration, from the equation of the surface whose partial area is being sought. This is illustrated in Ex. 2.

Ex. 1. Find the surface of the sphere whose equation is

$$x^2 + y^2 + z^2 = a^2.$$

Let O–ABC (Fig. 45) be one eighth of the sphere. In this case,

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z};$$

and hence $$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{a^2}{z^2} = \frac{a^2}{a^2 - x^2 - y^2}.$$

$$\text{Therefore,}\quad \text{area of surface } ABC = \int_{x=0}^{x=OA}\int_{y=0}^{y=SL}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$= a\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

$$= a\int_0^a\left[\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$= \frac{\pi a}{2}\int_0^a dx = \frac{\pi a^2}{2}.$$

Hence, area of all the surface of the sphere is $4\pi a^2$. (Compare Ex. 1, Art. 74.)

Ex. 2. The center of a sphere, whose radius is a, is on the surface of a right cylinder the radius of whose base is $\frac{1}{2}a$. Find the surface of the cylinder intercepted by the sphere. On taking the origin at the center of the sphere, an element of the cylinder for the z-axis and a diameter of a right section of the cylinder for the x-axis, the equation of the sphere will be

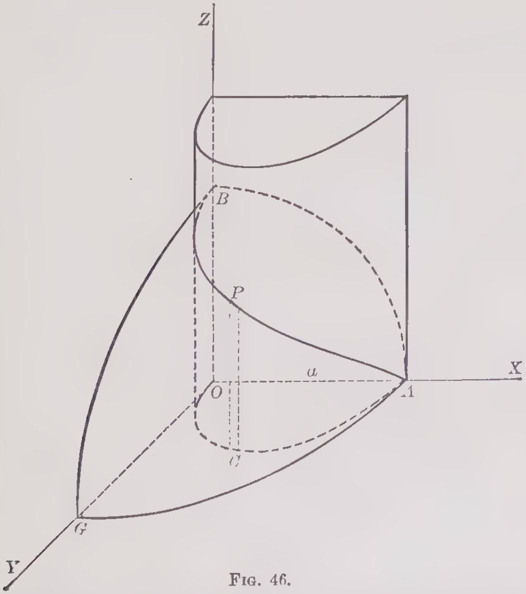

FIG. 46.

$$x^2+y^2+z^2=a^2,$$

and the equation of the cylinder, $x^2+y^2=ax$.

The area of the strip CP will first be found, and then the strips in the cylindrical surface $APBOCA$ will be summed. The element of surface in the strip CP is $dx\,dz$. Hence,

Cylindrical surface intercepted $= 4\,APBOCA$

$$= 4\int_{x=0}^{x=OA}\int_{z=0}^{z=CP}\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}} dx\,dz.$$

Since the surface required is on the cylinder, the partial derivatives must be derived from the second of the equations above. Hence,

$$\frac{\partial y}{\partial x}=\frac{a-2x}{2y},\quad \frac{\partial y}{\partial z}=0.$$

Also, $CP^2 = z^2 = a^2-(x^2+y^2)$, since P is on the sphere,

and hence, $= a^2 - ax,$ since P is on the cylinder.

Moreover, $OA = a$.

Therefore, the cylindrical surface intercepted

$$= 4\int_0^a\int_0^{\sqrt{a^2-ax}}\left[1+\left(\frac{a-2x}{2y}\right)^2\right]^{\frac{1}{2}} dx\,dz.$$

But on the cylinder, $y^2 = ax - x^2$. Hence, the intercepted cylindrical surface

$$= 2a\int_0^a\int_0^{\sqrt{a^2-ax}}\frac{dx\,dz}{\sqrt{ax-x^2}}$$

$$= 2a\int_0^a\frac{\sqrt{a^2-ax}}{\sqrt{ax-x^2}}\,dx = 2a\int_0^a\sqrt{\frac{a}{x}}\,dx$$

$$= 4a^2.$$

Ex. **3**. In the preceding example, find the surface of the sphere intercepted by the cylinder.

Ex. **4**. Find the area of the portion of the surface of the sphere

$$x^2+y^2+z^2 = 2ay$$

lying within the paraboloid

$$y = Ax^2 + Bz^2.$$

76. Mean values. The mean value of n quantities is the nth part of their sum. Let $\phi(x)$ be any continuous function of x, and let an interval, $b - a$, be divided into n parts, each equal to h. The mean value of the n quantities,

$$\phi(a),\ \phi(a+h),\ \phi(a+2h),\ \cdots,\ \phi(a+(n-1)h),$$

is
$$\frac{\phi(a)+\phi(a+h)+\phi(a+2h)+\cdots+\phi(a+\overline{n-1}h)}{n}.$$

Since $n = \dfrac{b-a}{h}$, this mean value is

$$\frac{\phi(a)h+\phi(a+h)h+\cdots+\phi(a+\overline{n-1}h)h}{b-a}.$$

Now suppose that x takes all the possible values, infinite in number, that are in the interval between a and b. Then, n is infinite, h is infinitesimal, and the number of terms in the last numerator is infinite. The sum of all these terms, by Art. 4, is expressed by

$$\int_a^b \phi(x)\,dx.$$

Hence, the mean value of all the values that a continuous function, $\phi(x)$, can take in the interval $b - a$ for x is

$$\frac{\int_a^b \phi(x)\,dx}{b-a}.$$

This is usually called the mean value of the function $\phi(x)$ over the *range* $b - a$. A geometric conception of the mean value was given in Art. 7 (*c*). A more general definition of mean value is given in Art. 77.

It is necessary to understand clearly the law according to which the successive values of the function are taken. Exs. 1, 2, Exs. 6, 7, and Exs. 12, 13, will serve to illustrate this remark.

Ex. 1. Find the mean velocity of a body when falling from rest, the velocities being taken at equal intervals of time.

In the case of a body falling from rest, $v = \frac{1}{2}gt$. Hence, calling V the mean velocity for a time t_1,

$$V = \frac{\displaystyle\int_{t=0}^{t=t_1} v\,dt}{t_1} = \frac{1}{t_1}\int_0^{t_1} \tfrac{1}{2} gt\,dt = \tfrac{1}{4} gt_1 = \tfrac{1}{2} v_1;$$

that is, the mean velocity is one half the final velocity.

Ex. 2. In the case of a body that falls from rest, find the mean of the velocities which the body has at equal intervals of space. It is known that, if s is the distance through which a body has fallen on starting from rest, and v is the velocity required,

$$v^2 = 2\,gs.$$

Hence, the mean velocity,

$$V = \frac{1}{s_1}\int_{s=0}^{s=s_1} \sqrt{2\,gs}\,ds = \tfrac{2}{3}\sqrt{2\,gs_1} = \tfrac{2}{3} v_1;$$

that is, the mean of the equal-distance velocities is equal to two thirds of the final velocity.

Ex. 3. Find the average value of the function $3x^2 + 5x - 7$ as x varies continuously from 1 to 4.

Ex. 4. Find the average value of the function $x^3 - 3x^2 + 2x - 1$ as x varies continuously from 0 to 3.

Ex. 5. Find the average ordinate drawn,

(*a*) in the curve, $y = x^2 + x + 1$ between the abscissas 2, 3;
(*b*) in the curve, $y = (x + 1)(x + 2)$ between the abscissas 1, 3;
(*c*) in the curve, $x^4 + ax^3 + a^2x^2 + b^2y = 0$ between the abscissas a, 0.

Ex. 6. Find the mean length of the ordinates of a semicircle (radius a), the ordinates being erected at equidistant intervals on the diameter.

Ex. 7. Find the mean length of the ordinates of a semicircle (radius a), the ordinates being drawn at equidistant intervals on the arc.

Ex. 8. Find the mean value of $\sin\theta$ as θ varies from 0 to $\frac{\pi}{2}$.

Ex. 9. Find the mean distance of the points on the circumference of a circle of radius a, from a fixed point on the circumference.

Ex. 10. Find the mean latitude of all the places north of the equator.

Ex. 11. A number n is divided at random into two parts. Find the mean value of their product.

Ex. 12. Show that the mean of the squares on the diameters of an ellipse that are drawn at equal angular intervals is equal to the rectangle contained by the major and minor axes.

Ex. 13. Show that the mean of the squares on the diameters of an ellipse that are drawn at points on the curve whose eccentric angles differ successively by equal amounts, is equal to one half the sum of the squares on the major and minor axes.

77. A more general definition of mean value. In Exs. 1, 3, 4, below, "the range" over which the function (in these cases, the distance of a point) varies, is a plane area. In Ex. 2, the range is a curvilinear area; and in Exs. 5, 6, it is a portion of space. The following may be taken for the definition of the mean value of a function, whatever the range may be:

$$\left\{\begin{array}{l}\textbf{The mean value of a}\\ \textbf{function throughout}\\ \textbf{any range}\end{array}\right\} = \frac{\sum\left\{\begin{array}{l}\textbf{(The value of the function for each}\\ \textbf{element of the range)} \times \textbf{(the ele-}\\ \textbf{ment of the range)}\end{array}\right\}}{\textbf{The range}},$$

in which the summation in the numerator is made throughout the whole of the range. The mean value considered in Art. 76 is merely a special case.

Ex. 1. Find the mean distance of a fixed point on the circumference of a circle of radius a from all points within the circle.

On taking the fixed point for the pole and the tangent thereat for the initial line, the value of the function (in this case the distance) at any point (r, θ) is r. The element of the range (in this case an area) at the point is $r\,d\theta\,dr$. This is shown in Fig. 47. Hence,

$$\text{the mean distance} = \frac{\int_0^{\pi}\int_0^{2a\sin\theta} r^2\,d\theta\,dr}{\int_0^{\pi}\int_0^{2a\sin\theta} r\,d\theta\,dr}$$

$$= \frac{\frac{8a^3}{3}\int_0^{\pi}\sin^3\theta\,d\theta}{\pi a^2}$$

$$= \frac{32\,a}{9\,\pi} = 1\cdot 132\,a.$$

(See Ex. 1, Art. 67.)

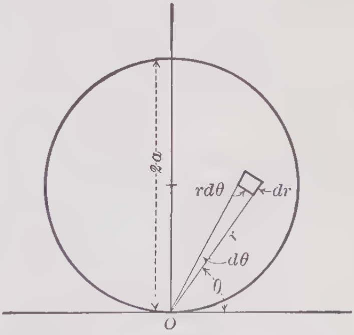

Fig. 47.

Ex. 2. Find the mean length of the ordinates drawn from all points on the curved surface of a hemisphere of radius a to its diametral plane.

Ex. 3. Find the mean length of the ordinates drawn from all the points of its diametral plane to the surface of the hemisphere of radius a.

Ex. 4. Find the mean square of the distance of a point within a given square (side $= 2a$) from the center of the square.

Ex. 5. Find the mean distance of all the points within a sphere of radius a from a given point on the surface.

Ex. 6. Find the mean distance of all the points within a sphere of radius a from the center.

EXAMPLES ON CHAPTER IX.

1. Find the volume of a sphere of radius a by means of a single integration. (Suppose that the sphere is made up of infinitely thin concentric spherical shells of thickness dr. The volume of each shell $= 4\pi r^2 dr$; hence volume of sphere $= 4\pi\int_0^a r^2 dr = \frac{4}{3}\pi a^3$.

2. Find the volume and surface generated by revolving about the y-axis the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

3. Find the surface generated by the revolution about the y-axis of the arc of the parabola $y^2 = 4ax$ from the origin to the point (x, y).

4. Find the volume generated by revolving the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$ about its asymptote.

5. Find the convex surface of the cone generated by revolving about the x-axis the line joining the origin and the point (a, b).

6. Find the surface of the torus generated by revolving about the x-axis the circle $x^2 + (y - b)^2 = a^2$.

7. On the double ordinates of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and in planes perpendicular to that of the ellipse, isosceles triangles of vertical angle 2α are described. Find the volume of the surface thus constructed.

8. Two cylinders of equal altitude h have a circle of radius a for their common upper base. Their lower bases are tangent to each other. Find the volume common to the two cylinders.

9. Find the volume inclosed by two right circular cylinders of equal radius a whose axes intersect at right angles. Also, find the surface of one intercepted by the other.

10. Find the volume of the solid contained between

the paraboloid of revolution, $x^2 + y^2 = az$;
the cylinder, $x^2 + y^2 = 2ax$;
and the plane, $z = 0$.

11. Find the entire volume bounded by the surface

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

12. An arc of a circle revolves about its chord. Find the volume and surface of the solid generated, a being the radius, and 2α the angular measure of the arc.

13. A cycloid revolves about the tangent at the vertex. Find the surface and volume of the solid generated.

14. A cycloid revolves about its base. Find the area of the surface generated.

15. A cycloid revolves about its axis. Find the surface and volume generated.

16. A quadrant of an ellipse revolves round a tangent at the end of the minor axis of the ellipse. Find the volume of the solid generated.

17. If b be the radius of the middle section of a cask, a the radius of either end, and h its length, find the volume of the cask, assuming that the generating curve is an arc of a parabola.

18. Find the length of the curve $9ay^2 = x(x - 3a)^2$ from $x = 0$ to $x = 3a$.

19. Find the length of the logarithmic curve $y = ca^x$.

20. Find the length of an arch of the epicycloid,

$$x = (a + b)\cos\theta - b\cos\frac{a + b}{b}\theta,$$

$$y = (a + b)\sin\theta - b\sin\frac{a + b}{b}\theta.$$

21. Find the length of an arc of the evolute of the parabola $y^2 = 4px$, namely,

$$27py^2 = 4(x - 2p)^3$$

from the point where $x = 2p$ to the point where $x = 3p$. Also, find the length of arc of the preceding curve from the cusp $(x = 2p)$ to the point where it intersects the parabola (at the point for which $x = 8p$).

22. Find the length of the arc of the curve $y = \log \sin x$ between $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$.

23. Find the length of the arc of the evolute of the circle,

$$x = a(\cos\theta + \theta\sin\theta),$$
$$y = a(\sin\theta - \theta\cos\theta),$$

from $\theta = 0$ to $\theta = \alpha$.

24. Find the entire length of the curve $r = a\sin^3\frac{\theta}{3}$.

25. Find the length of arc of the spiral $r = m\theta^2$, from $\theta = 0$ to $\theta = \theta_1$.

CHAPTER X

APPLICATIONS TO MECHANICS

78. Mass and density. This chapter is introduced for the purpose of giving the student further examples of the application of the fundamental principle of the integral calculus, and of affording him additional opportunity for practice in integration. The definitions of mechanics that are required in what follows are merely stated, but are not discussed. They will be familiar to those who have had the advantage of an elementary course in that subject. Other readers can only assume these definitions as *data* for problems in integration.

Mass. The mass of a body is usually defined as "the quantity of matter which it contains," and is specified in terms of the mass of a standard body. In English-speaking countries, for ordinary purposes, the standard mass is a certain bar of platinum marked "P.S. 1844. 1 lb.," which is called the "imperial standard pound avoirdupois," and is preserved at the Office of the Exchequer in London. Any mass equal to this standard mass is then a unit of mass. For scientific purposes in general, and in countries where the metric system is adopted, the standard of mass is the "kilogramme des archives," a bar of platinum kept in the Palais des Archives in Paris. A mass equal to one thousandth of this standard is then the unit of mass; this unit is called the gram. The mass of a body should not be confounded with its weight. The weight of a body depends upon its distance from the center of the earth, but its mass is independent of its position.

Density. The *mean density* of a body is the quotient of its mass by its volume. *The density at a given point* of a body is the quotient of the mass of an infinitesimal portion of the body surrounding the given point by the volume of the same portion. If the density of a body is the same at all of its points, it is said to be *homogeneous.*

79. Center of mass. Suppose that there is a system of particles whose masses are m_1, m_2, $\cdots$, and whose distances from a given plane are r_1, r_2, $\cdots$. There is a point whose distance D from the given plane, no matter what plane it may be, always satisfies the condition

$$D = \frac{m_1 r_1 + m_2 r_2 + \cdots}{m_1 + m_2 + \cdots}; \tag{1}$$

that is, the condition

$$D = \frac{\Sigma mr}{\Sigma m}. \tag{2}$$

This point is *the center of mass.* In other words, the distance of the center of mass of a system of particles from any plane is equal to the sum of the products of the masses of the particles into their distances from the plane, divided by the total mass of the system. The center of gravity of a body, when it has one, coincides with the center of mass, and the former term is often used for the latter. The position of a point is known if its distances from three planes, no two of which are parallel to each other, are known. If x_1, y_1, z_1, x_2, y_2, z_2, $\cdots$ are the coördinates of the particles of mass m_1, m_2, $\cdots$, respectively, and if $\bar{x}$, y, $\bar{z}$ are the coördinates of their center of mass, then by (2)

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}; \quad \bar{y} = \frac{\Sigma my}{\Sigma m}; \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}. \tag{3}$$

In cases in which matter is continuously distributed, — for example, as in a bar, a solid sphere, a cylindrical shell, etc., — the matter in the bar, sphere, shell, etc., may be supposed to be divided into small portions whose masses are Δm_1, Δm_2, $\cdots$. If a point be taken in each of these elements, and the distances of these points from a fixed plane be r_1, r_2, $\cdots$, then the smaller the portions Δm the more nearly do they come to being particles with distances $r_1, r_2, \cdots$, from the fixed plane. Ultimately, therefore, the distance of their center of mass from the plane is given by

$$D = \text{limit}_{\Delta m \doteq 0} \frac{\Sigma r \, \Delta m}{\Sigma \, \Delta m}.$$

In accordance with the first definition of integration this is written,

$$D = \frac{\int r \, dm}{\int dm}.$$

Therefore, in the case of any continuous distribution of matter, the coördinates $\bar{x}, \bar{y}, \bar{z}$, of the center of mass are given by

$$\bar{x} = \frac{\int x \, dm}{\int dm}; \quad \bar{y} = \frac{\int y \, dm}{\int dm}; \quad \bar{z} = \frac{\int z \, dm}{\int dm}. \tag{4}$$

If ρ be the density at any point of a body, and dv an infinitesimal volume about the point,

$$dm = \rho \, dv,$$

$$\text{the total mass} = \int \rho \, dv,$$

and formulæ (4) become

$$\bar{x} = \frac{\int \rho x \, dv}{\int \rho \, dv}; \quad \bar{y} = \frac{\int \rho y \, dv}{\int \rho \, dv}; \quad \bar{z} = \frac{\int \rho z \, dv}{\int \rho \, dv}. \tag{5}$$

The density ρ usually varies from point to point of a body, and it is generally expressed as some function of the position of the point. If the body is homogeneous, ρ is constant and can be removed from formulæ (5) by cancellation. If $\bar{\rho}$ be the mean density of a non-homogeneous body, then, by the definition in Art. 78,

$$\rho = \frac{\int \rho \, dv}{\int dv}. \tag{6}$$

If matter be supposed to be continuously distributed along a line or curve making, as it were, a wire of infinitesimal cross-section, or so thinly laid upon a surface, curvilinear or plane, that the thickness of the layer may be neglected, the term "mass-center" can also be used with reference to lines and curves, surfaces and plane areas. If Δs, ΔS, ΔA are small elements of a line or curve, a curvilinear surface, and a plane area respectively, and ρ is the linear density or the surface density, the coördinates of the mass-center of an arc, surface, or plane area are obtained from formulæ (**5**) on the substitution of ds, dS, dA respectively for dV. Expressions for these differentials have already been obtained in the preceding articles. The mean linear and surface density can be obtained by making these substitutions in formula (**6**).

Ex. 1. Find the total mass and the mean density of a very thin plate which is the first quadrant of the circle whose equation is $x^2 + y^2 = a^2$, and whose density varies at each point as xy.

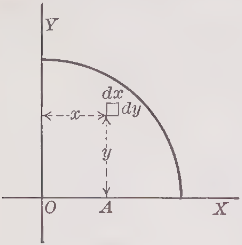

Fig. 48.

If ρ denote the density, then by the given condition,

$$\rho \propto xy;$$

that is, $$\rho = kxy,$$

in which k is some constant.

If M denote the total mass of the quadrant, and dm denote the mass of an infinitesimal rectangle about any point,

$$M = \int dm = \int \rho \, dA$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} k\,xy\,dx\,dy$$

$$= \tfrac{1}{8} ka^4.$$

If $\bar{\rho}$ be the mean density, it follows from the definition that,

$$\bar{\rho} = \frac{\int \rho \, dA}{\int dA} = \frac{\frac{1}{8} ka^4}{\frac{1}{4} \pi a^2} = \frac{ka^2}{2\pi}.$$

Ex. 2. Find the center of mass of the thin plate described in Ex. 1.

Here,
$$\bar{x} = \frac{\int \rho x \, dv}{\int \rho \, dv} = \frac{\int kxy\, x\, dA}{M}$$

$$= \frac{k\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dx\, dy}{M}$$

$$= \frac{\frac{1}{15} ka^5}{\frac{1}{8} ka^4} = \frac{8}{15} a.$$

Similarly,
$$\bar{y} = \frac{8}{15} a.$$

Ex. 3. Find the mass-center for a thin hemispherical shell, radius a, whose density at each point of the surface varies as the distance y from the plane of the rim.

Let the hemisphere be described by revolving the semicircle of radius a and center O about the y-axis OY, which is at right angles to the diameter, the point O being taken for the origin of coördinates. Let P, whose coördinates are x, y, be any point on the semicircle, and draw PM, PN at right angles to the axes of x and y respectively. Join OP, and denote the angle NOP by θ.

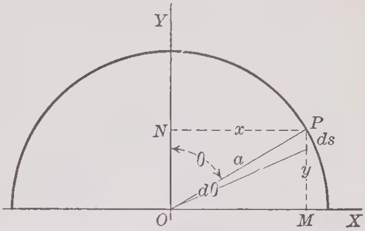

Fig. 49.

At the point P, $\quad y = a\cos\theta$;

also at P, $\quad \rho \propto y$,

that is, $\quad \rho = ka\cos\theta$,

in which k is some constant. The infinitesimal arc of length ds at P describes a zone about OY whose area is given by

$$dS = 2\pi NP\, ds.$$

or, since $ds = a d\theta$,
$$= 2\pi a \sin\theta \cdot a\, d\theta.$$

The symmetry of the figure shows that

$$\bar{x} = 0.$$

Also,
$$\bar{y} = \frac{\int \rho y \, dS}{\int \rho \, dS} = \frac{2\pi ka^4 \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta}{2\pi ka^3 \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta}$$

$$= \frac{2}{3} a.$$

Hence, the center of mass is at the point $(0, \frac{2}{3} a)$.

Ex. 4. Find the center of mass of a right circular cone of height h, which is generated by the revolution of the line $y = ax$ about the x-axis, when the density of each infinitely thin cross-section varies as its distance from the vertex.

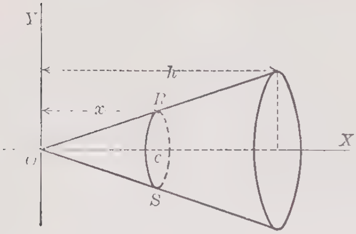

FIG. 50.

Symmetry shows that the center of mass is in the x-axis. Suppose that a very thin plate RS is taken which cuts the axis of the cone at right angles at C at a distance x from the vertex.

The radius CR of the cross-section $= ax.$

The density of this thin plate, $\rho = kx.$

The volume of the thin plate, $dV = \pi\,\overline{CR}^2\,dx$

$$= \pi a^2 x^2\,dx.$$

Hence,

$$\bar{x} = \frac{\int \rho x\,dV}{\int \rho\,dV} = \frac{k\pi a^2 \int_0^h x^4\,dx}{k\pi a^2 \int_0^h x^3\,dx}$$

$$= \tfrac{4}{5}h.$$

Ex. 5. Find the mean density of the cone described in Ex. 4.

Ex. 6. Find the mass-center of the surface of the cone in Ex. 4.

Ex. 7. Find the mass-center of the cone generated in Ex. 4, and the mass-center of its convex surface when the density is uniform.

Ex. 8. Find the mass-center of a quadrantal arc of the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Ex. 9. Find the mass-center of the convex surface of a hemisphere of radius 10.

Ex. 10. The quadrant of a circle of radius a revolves about the tangent at one extremity; prove that the distance of the mass-center of the generated curved surface from the vertex is $.876\,a$.

Ex. 11. Find the mass-center of the semicircle of $x^2 + y^2 = a^2$ on the right of the y-axis.

Ex. 12. Find the mass, the mean density, and the mass-center of the semicircle in Ex. 11 when the density varies as the distance from the diameter.

Ex. 13. Find the mass-center of a circular sector of angle 2α, taking the origin at the center, and the x-axis along the bisector of the angle.

Ex. 14. Find the mass-center of the first quadrant of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Ex. 15. Find the mass-center of the area between the parabola $y^2 = 4\,ax$, and: (a) the double ordinate for $x = h$; (b) the ordinate for $x = h$ and x-axis.

Ex. 16. Show that the mass-center of the circular spandril formed by a quadrant of a circle of radius a and the tangents at its extremities is at a distance $.2234\,a$ from either tangent.

Ex. 17. Find the mass-center of a quadrant of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 18. Find the mass-center of the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the axes.

Ex. 19. Find the mass-center of the area between the cissoid $y^2 = \dfrac{x^3}{a - x}$ and its asymptote.

Ex. 20. Find the mass-center of the cardioid $r = 2\,a\,(1 - \cos\theta)$.

Ex. 21. Find the center of mass of the solid paraboloid generated by the revolution of $y^2 = 4\,ax$ about the x-axis.

Ex. 22. Show that the center of mass of a solid hemisphere of uniform density and radius a, is at a distance $\frac{3}{8}\,a$ from the plane of the base.

Ex. 23. Show that the center of mass of a solid hemisphere, radius a, in which the density varies as the distance from the diametral plane is at a distance $\frac{8}{15}\,a$ from this plane. Also show that the mean density of this hemisphere is equal to the density at a distance $\frac{3}{8}\,a$ from the base.

Ex. 24. Find the center of mass of a solid hemisphere, radius a, in which the density varies as the distance from the center of the sphere.

Ex. 25. Find the center of mass of the solid generated by the revolution of the cardioid $r = 2\,a\,(1 - \cos\theta)$ about its axis.

80. Moment of inertia. Radius of gyration. If in any system of particles the mass of each particle be multiplied by the square of its distance from a given line, the sum of the products thus obtained is called the *moment of inertia* of the system about that line. Thus, if m_1, m_2, $\cdots$, be the masses of the several particles,

$r_1, r_2, \cdots$, their distances from the line, and I denote the moment of inertia,

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots;$$

that is, $$I = \Sigma m r^2. \tag{1}$$

In any case in which matter is continuously distributed, as in a solid cylinder, a shell, etc., the matter may be supposed to be divided into small portions, $\Delta m_1, \Delta m_2, \cdots$. By reasoning similar to that employed in the last article, it can be shown that

$$I = \int r^2 dm.$$

If matter be supposed to be distributed uniformly along a line or curve, or upon a curvilinear surface or a plane area, the term "moment of inertia" can also be used in reference to curves, surfaces, and plane areas.

Let M denote the total mass of a body, namely $\int dm$, and I its moment of inertia about a given line or axis. If k satisfies the equation

$$Mk^2 = I;$$

that is, if $$k^2 = \frac{I}{M} = \frac{\int r^2 dm}{\int dm},$$

k is called the *radius of gyration* of the body about the given axis.

Ex. 1. Find the moment of inertia of a rectangle of uniform density, whose sides have the lengths b, d about a line which passes through the center of the rectangle and is parallel to the sides of length b.

The density per unit of area will be represented by unity. Let the axes of x and y be taken parallel to the sides of the rectangle, the origin being at the center, and let

$$AB = b, \quad BC = d.$$

Then $I = \int y^2\, dA$

$$= \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2\, dy\, dx$$

$$\frac{bd^3}{12}.$$

This moment of inertia is important in calculations on beams. Since

the mass of the rectangle $= bd$,

$$k^2 = \frac{I}{M} = \frac{d^2}{12}.$$

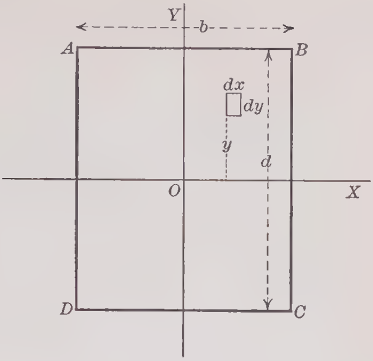

Fig. 51.

Ex. 2. Find the moment of inertia of a very thin circular plate of uniform density of radius a about an axis through its center and perpendicular to its plane.

Taking the density as unity per unit of area,

$$I = \int r^2\, dm = \int r^2\, dA$$

$$= \int_0^a \int_0^{2\pi} r^2 \cdot r\, dr\, d\theta$$

$$= \frac{\pi a^4}{2}.$$

Also, $$k^2 = \frac{I}{M} = \frac{\frac{\pi a^4}{2}}{\pi a^2} = \frac{a^2}{2}.$$

Ex. 3. Find the moment of inertia about its axis of a right circular cone of height h and base of radius b, the density being uniform, and m being the mass per unit of volume.

The moment of inertia is equal to the sum of the moments of inertia of very thin transverse plates like RS. If

$$OC = x,$$

then, by similar triangles,

$$RC = \frac{b}{h}(h - x).$$

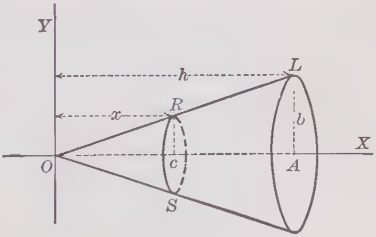

Fig. 52.

Hence, if dI denote the moment of inertia of the plate RS of thickness dx, by Ex. 2,

$$dI = \frac{m\pi b^4}{2\,h^4}(h - x)^4\,dx.$$

Therefore, for the whole cone,

$$I = \frac{\pi b^4\,m}{2\,h^4}\int_0^h (h - x)^4\,dx$$

$$= \frac{m\pi}{10}\,b^4h.$$

Also,

$$k^2 = \frac{I}{M} = \frac{I}{m\,V} = \frac{\frac{1}{10}\,m\pi b^4h}{\frac{1}{3}\,m\pi b^2h}$$

$$= \tfrac{3}{10}\,b^2.$$

Ex. **4.** Find the radius of gyration of a uniform circular wire about its diameter.

Ex. **5.** Find the moment of inertia of the triangle formed by the axes and a line whose intercepts are a and b, about an axis which passes through the origin, and is at right angles to the plane of the triangle.

Ex. **6.** Find the radius of gyration about its line of symmetry of an isosceles triangle of base $2\,a$ and altitude h.

Ex. **7.** Find the moment of inertia about the x-axis of the area between the line and the parabola which both pass through the origin and the point (a, b), the axis of the parabola being along the x-axis.

Ex. **8.** Find the moments of inertia of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$: (a) about the x-axis; (b) about the y-axis; (c) about an axis that passes through the center of the ellipse and is perpendicular to the plane of the ellipse Apply the results to the circle $x^2 + y^2 = a^2$

Ex. **9.** Find the moment of inertia of the thin plate in Ex. 1, Art. 79, about the x-axis.

Ex. **10.** Find the moment of inertia of a homogeneous ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

about the x-axis.

Ex. **11.** Find the moment of inertia of the surface of a sphere of radius a about a diameter, m being the mass per unit of surface.

Ex. **12.** Find the moment of inertia of a solid homogeneous sphere of radius a about a diameter, m being the mass per unit of volume.

Ex. **13.** Find the moment of inertia of the semicircular plate described in Ex. 12, Art. 79, about the diameter.

Ex. **14.** Find the moment of inertia, and the radius of gyration about its axis, of a homogeneous right circular cylinder of length l and radius R, m being the mass per unit of volume. Also about a diameter of one end.

CHAPTER XI

APPROXIMATE INTEGRATION. INTEGRATION BY MEANS OF SERIES. INTEGRATION BY MEANS OF THE MEASUREMENT OF AREAS

81. Approximate integration. It was remarked in Arts. 4, 8 that in most cases in which a differential $f(x)\,dx$ is given it is not possible to find the anti-differential. In some of these cases, however, an expression can be found that will approximately represent the indefinite integral $\int f(x)\,dx$. Even if this cannot be done, it is often possible to determine a value that will very nearly be that of the definite integral $\int_a^b f(x)\,dx$. Art. 82 explains a method, that of integration in series, by means of which an indefinite integral may be expressed as a function of x in the form of a series that contains an infinite number of terms. An important application of this method to another problem is given in Art. 83. Arts. 84–87 set forth a method, that of measurement of areas, which reduces the evaluation of a definite integral to a mere matter of careful computation. In this connection several formulæ for the approximate determination of areas are necessarily considered.

82. Integration in series. When the indefinite integral of a given function, $f(x)\,dx$, cannot be found by any of the means thus far considered, one of the most usual and most fruitful methods employed is the following: The function $f(x)$ is developed in a series in ascending or descending powers of x. If this series is convergent within certain limits for x, the series obtained by integrating it term by term is also convergent within the same limits.* The greater the number of terms taken the more nearly will the new series represent $\int f(x)\,dx$.

Ex. 1. Find $\displaystyle\int \frac{dx}{(1+x^5)^{\frac{2}{3}}}$.

By the binomial theorem,

$$(1)\quad \frac{1}{(1+x^5)^{\frac{2}{3}}} = (1+x^5)^{-\frac{2}{3}} = 1 - \frac{2}{1}\cdot\frac{x^5}{3} + \frac{2\cdot 5}{1\cdot 2}\cdot\frac{x^{10}}{3^2} - \frac{2\cdot 5\cdot 8}{1\cdot 2\cdot 3}\cdot\frac{x^{15}}{3^3} + \cdots.$$

The second member is convergent for values of x between $+1$ and -1. Integration of both members of (1) gives

$$(2)\quad \int \frac{dx}{(1+x^5)^{\frac{2}{3}}} = c + x - \frac{2}{1}\cdot\frac{x^6}{3\cdot 6} + \frac{2\cdot 5}{1\cdot 2}\cdot\frac{x^{11}}{3^2\cdot 11} - \frac{2\cdot 5\cdot 8}{1\cdot 2\cdot 3}\cdot\frac{x^{16}}{3^3\cdot 16} + \cdots.$$

The second member represents the required integral for values of x between $+1$ and -1. It follows from (2) that

$$\int_0^1 \frac{dx}{(1+x^5)^{\frac{2}{3}}} = 1 - \frac{2}{3\cdot 6} + \frac{2\cdot 5}{1\cdot 2}\cdot\frac{1}{3^2\cdot 11} - \frac{2\cdot 5\cdot 8}{1\cdot 2\cdot 3}\cdot\frac{1}{3^3\cdot 16} + \cdots.$$

Ex. 2. Find $\int e^{x^2}dx$.

Since
$$e^z = 1 + z + \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \cdots,$$

$$(1)\quad e^{x^2} = 1 + x^2 + \frac{x^4}{1\cdot 2} + \frac{x^6}{1\cdot 2\cdot 3} + \cdots,$$

which is convergent for all finite values of x.

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* Suppose that

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n + \cdots. \qquad (1)$$

Then
$$\int f(x)\,dx = a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \cdots + \frac{a_{n-1}x^n}{n} + \frac{a_nx^{n+1}}{n+1} + \cdots. \qquad (2)$$

The series in (1) is convergent when $\dfrac{a_nx}{a_{n-1}}$ is less than unity for all values of n beyond some finite number. The series in (2) is convergent when $\dfrac{n}{n+1}\,\dfrac{a_nx}{a_{n-1}}$, and therefore when $\dfrac{a_nx}{a_{n-1}}$, is less than unity for all values of n beyond a certain number. Since the convergency of both series depends upon the same condition, the second series is convergent when the first is convergent.

Integration of both members of (1) gives

$$(2)\quad \int e^{x^2}\,dx = c + x + \frac{x^3}{3} + \frac{x^5}{1\cdot 2\cdot 5} + \frac{x^7}{1\cdot 2\cdot 3\cdot 7} + \frac{x^9}{1\cdot 2\cdot 3\cdot 4\cdot 9} + \cdots.$$

The second member represents the required integral for values of x between $+1$ and -1. It follows from (2) that

$$\int_{-1}^{+1} e^{x^2}\,dx = 2\left(1 + \frac{1}{3} + \frac{1}{1\cdot 2\cdot 5} + \frac{1}{1\cdot 2\cdot 3\cdot 7} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 9} + \text{etc.}\right)$$

Ex. 3. $\int \frac{dx}{\sqrt{1+x^4}}$.

Ex. 4. $\int \frac{dx}{\sqrt{1-x^5}}$.

Ex. 5. $\int x^{\frac{1}{3}}\sqrt{1-x^2}\,dx$.

Ex. 6. $\int x^2\sqrt{1-x^2}\,dx$.

(Compare Ex. 28, page 98.)

Ex. 7. $\int (1+cx^n)^{\frac{p}{q}} x^{m-1}\,dx$.

Ex. 8. $\int \frac{dx}{\sqrt{\sin x}}$. (Put $\sin x = z$.)

Ex. 9. $\int \frac{e^x\,dx}{x}$.

Ex. 10. $\int \frac{e^{mx}}{x}\,dx$.

Ex. 11. $\int \frac{\sin x}{x}\,dx$.

83. Expansion of functions by means of integration in series. A function can be developed in series by means of the method described in the last article if the expansion of its derivative is known. The series which represents the function is obtained by integrating the series which represents the derivative, and determining the value of the constant of integration.

Ex. 1. Expand $\tan^{-1} x$ in a series of ascending powers of x.

Differentiation and division give

$$d\cdot\tan^{-1}x = \frac{dx}{1+x^2} = (1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} - \cdots)\,dx,$$

which is convergent when x lies between -1 and $+1$.

Integrating,

$$\tan^{-1}x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} - \cdots.$$

The substitution of 0 for x gives

$$m\pi = c,$$

m being an integer; and hence,

$$\tan^{-1}x = m\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

This series* can be employed for values of x between -1 and $+1$. It can be used for computing the value of π. For, on putting $x = \frac{1}{\sqrt{3}}$ therein, it is found that

$$\tan^{-1}\frac{1}{\sqrt{3}} = m\pi + \frac{\pi}{6} = m\pi + \frac{1}{\sqrt{3}}\left(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \cdots\right);$$

whence, $$\pi = 2\sqrt{3}\left(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \cdots\right).$$

Ex. **2.** Expand † $\sin^{-1}x$ in a series in x; and compute the value of π by putting $x = \frac{1}{2}$.

Ex. **3.** Derive ‡ $\int e^{-x^2}dx = 1 - \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 2\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdot 7} + \cdots$, which is convergent for all finite values of x.

Ex. **4.** Show that

$$\log(a + x) = \log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \cdots \text{ when } |x| < 1;$$

and that $\log(a + x) = \log a + \frac{a}{x} - \frac{a^2}{2x^2} + \frac{a^3}{3x^3} - \frac{a^4}{4x^4} + \cdots$ when $|x| > 1$.

The symbol $|x|$ denotes the absolute value of x.

Ex. **5.** Derive series for $\log(1 + x)$, $\log(1 - x)$, $\log 2$, $\log 9$.

Ex. **6.** Develop $\log(x + \sqrt{1 + x^2})$ in a series by integrating $(1 + x^2)^{-\frac{1}{2}}dx$.

Ex. **7.** § Show that

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \frac{\pi}{2}\left[1 + (\tfrac{1}{2}k)^2 + \left(\frac{1\cdot 3}{2\cdot 4}k^2\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^3\right)^2 + \cdots \right.$$
$$\left. + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}k^n\right)^2 + \cdots\right],$$

k^2 being less than unity. (See Ex. 9, Art. 46.)

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* It is usually called Gregory's series, after its discoverer, James Gregory (1638–1675). It was found also by Leibniz (1646–1716).

† This expansion is due to Newton (1642–1727), and, by means of it, he computed the value of π.

‡ This integral is often met in the theory of probabilities, and in certain questions in physics. For the evaluation of $\int_0^x e^{-x^2}dx$ when x is greater than unity, see Laurent, *Cours d'Analyse*, t. III., § IV., p. 284. For the derivation of $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$, see Williamson, *Integral Calculus*, Ex. 4, Art. 116.

§ This integral is called the "elliptic integral of the first kind." It received the name elliptic integral from its similarity to the integral in Ex. 8, which represents the length of a quadrantal arc of an ellipse, and is known as "the elliptic integral of the second kind." The integral of the first and second kind are usually denoted by $F(k, \phi)$, $E(k, \phi)$, respectively. These names and symbols were given by Legendre (1752–1833).

Ex. 8. Show that

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\phi}\,d\phi = \frac{\pi}{2}\left[1 - \frac{1}{1}\left(\frac{1}{2}k\right)^2 - \frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}k^2\right)^2 - \frac{1}{5}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^3\right)^2 - \cdots - \frac{1}{2n-1}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots n}k^n\right)^2 - \cdots\right].$$

(See Ex. 9, Art. 46; Ex. 7, Art. 71.)

Ex. 9. Deduce the value of π by means of the series in Ex. 1, it being known that

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5}\tan^{-1}\frac{1}{239}.$$

84. Evaluation of definite integrals by the measurement of areas. It has been seen in Arts 4, 6, that the definite integral $\int_a^b f(x)\,dx$, may be graphically represented by the area included by the curve whose equation is $y = f(x)$, the axis of x, and the ordinates for which $x = a$, $x = b$; and it has been observed that the evaluation of the integral is equivalent to the measurement of this area. The numerical value of the integral $\int_a^b f(x)\,dx$, which is also the same as that of the area just described, has been obtained up to this point, by finding the anti-differential of $f(x)\,dx$, say $\phi(x)$, substituting b and a for x therein, and calculating $\phi(b) - \phi(a)$. But when it is not possible to find the anti-differential of $f(x)\,dx$, recourse must be had to other methods.

While, on the one hand, as already shown, areas may be determined by evaluating definite integrals, on the other hand, definite integrals may be evaluated by measuring areas. If the anti-differential of $f(x)\,dx$ is unknown, the value of $\int_a^b f(x)\,dx$ can be found in the following way. Plot the curve $y = f(x)$ from $x = a$ to $x = b$, erect the ordinates for which $x = a$, $x = b$, and measure the area bounded by the curve, the axis of x, and these ordinates. There are several rules or formulæ for determining areas of this kind. The degree of approximation to absolute correctness depends in general only on the patience of the calculator. These formulæ, some of which are usually given in manuals for engineers, are called "formulæ for the approximate determination of areas," or "formulæ for approximate quadrature." They may be given the more general title, "*formulæ for approximate integration.*" The two rules most frequently employed, namely, the trapezoidal rule and Simpson's one-third rule, are discussed in Arts. 85, 86, and a rule deduced from them is given in Art. 87. Other rules are given in the Appendix.*

It should be observed that only a numerical result is obtained by means of these rules. The knowledge of the value of the definite integral $\int_a^b f(x)\,dx$ thus calculated does not give any clue whatever to the expression of the indefinite integral $\int f(x)\,dx$ as a function of x. If the indefinite integral $\int f(x)\,dx$ has been found in the form of a series which is convergent for values of x between a and b, the value of the definite integral $\int_a^b f(x)\,dx$, can be found as accurately as one pleases by taking a sufficiently large number of terms. Illustrations of this remark have been given in Exs. 1, 2, Art. 82, and in Exs. 1, 2, 7, 8, Art. 83.

85. The trapezoidal rule. Let AK be a portion of a curve whose equation may or may not be known; and let LA, TK, be drawn at right angles to the line OX. It is required to find the area $AKTL$ contained between the curve AK, the line LT, and the perpendiculars LA, KT.

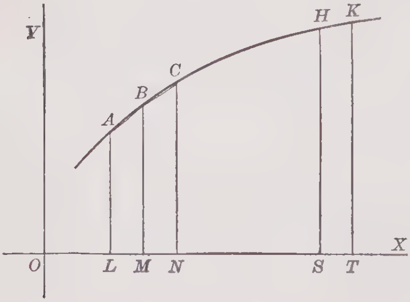

FIG. 53.

Divide LT into n parts, each equal to h, and at the points of division erect the perpendiculars MB, NC, $\cdots$, SH. Draw the chords AB, BC, $\cdots$, HK. A rule for finding the area of $LAKT$ will now be derived by substituting the sum of the trapezoidal areas, AM, BN, $\cdots$, HT, for the curvilinear area $LAKT$; that is, by substituting the boundary made up of the chords $AB, BC, \cdots, HK$, for the curved line AK. On adding the trapezoidal areas beginning at the left there is obtained,

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* See Note E.

$$\begin{aligned}\text{area} &= \frac{h}{2}(AL+BM)+\frac{h}{2}(BM+CN)+\cdots+\frac{h}{2}(HS+KT)\\ &= \frac{h}{2}(AL+2\,BM+2\,CN+\cdots+2\,HS+KT)\\ &= h\left(\tfrac{1}{2}+1+1+\cdots+1+1+\tfrac{1}{2}\right),\end{aligned}$$

on writing merely the coefficients of the successive ordinates. This mode of writing will be used also in the rules which follow. The greater the number of parts into which LT is divided, the nearer will the total area of the trapezoids be to the area required.

If the equation of the curve is $y=f(x)$, the axes being as in the figure, and $OL=a$, $OT=b$, the lengths of the successive ordinates beginning with LA are $f(a)$, $f(a+h)$, $f(a+2\,h), \cdots$, $f(b-h)$, $f(b)$. If LT is divided into n equal parts, $h=\dfrac{b-a}{n}$, and hence, approximately,

$$\begin{aligned}\int_a^b f(x)\,dx = \frac{b-a}{2n}\Big\{ f(a)+2f\left(a+\frac{b-a}{n}\right)\\ +2f\left(a+\frac{2(b-a)}{n}\right)+\cdots+2f(b-h)+f(b)\Big\}.\end{aligned}$$

Ex. 1. Evaluate $\int_0^{10} x^2\,dx$ by this method, taking unit intervals.

By the given condition, $h=1$; and hence, $n=10$. The successive ordinates, since $f(x)=x^2$, are 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. Hence, approximately,

$$\begin{aligned}\int_0^{10} x^2\,dx &= \tfrac{10}{20}\{0+100+2(1+4+9+16+25+36+49+64+81)\};\\ &= 335.\end{aligned}$$

The true value of the integral is $\left[\dfrac{x^3}{3}\right]_0^{10}$, that is, $333\frac{1}{3}$. Had the interval 0 to 10 been divided into more than 10 equal parts, the approximation to the true value would have been closer.

Ex. 2. Show that the approximate value obtained for the above integral, by making 20 equal intervals, is $333\frac{3}{4}$.

Ex. 3. Show that the approximate value of $\int_2^{12} \log_{10} x\, dx$, unit intervals being taken, is 7.990231.

86. The parabolic or Simpson's* one-third rule. The parabolic rule for approximating to the value of the area $LAKT$ is derived by substituting parabolic arcs through ABC, CDE, $\cdots$, GHK, for the arcs of the given curve passing through these points, the axes of the parabolas being vertical, and then summing the areas of the parabolic sections, $LABCN$, $NCDEP$, $\cdots$, $RGHKT$, which are thus formed. A parabolic arc, as CDE, will more nearly coincide with the given curve through CDE, than will the chords CD, DE. For the purposes of this rule, n the number of equal parts into which LT is divided must be even, since a parabolic strip is substituted for each of the consecutive pairs of trapezoidal strips; for example, $NCDEP$ for $ND + DP$.

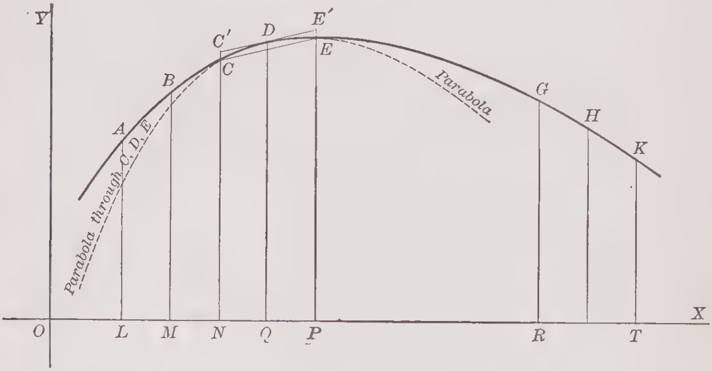

FIG. 54.

The area of one of the parabolic strips, say $NCDEP$ will first be found. Through D draw $C'E'$ parallel to the chord CE, and produce NC, PE to meet $C'E'$ in C', E'.

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* Thomas Simpson (1710–1761).

The parabolic strip $NCDEP =$ trapezoid $NCEP +$ parabolic segment CDE.

The parabolic segment $CDE =$ two thirds of its circumscribing parallelogram $CC'E'E$.

Hence, the parabolic strip

$$\begin{aligned} NCDEP &= NP[\tfrac{1}{2}(NC + PE) + \tfrac{2}{3}\{QD - \tfrac{1}{2}(NC + PE)\}] \\ &= 2h(\tfrac{1}{6}NC + \tfrac{2}{3}QD + \tfrac{1}{6}PE) \\ &= \frac{h}{3}(NC + 4QD + PE). \end{aligned}$$

Application of the latter formula to each of the parabolic strips in order beginning with the first on the left, and addition, gives, approximately,

$$\textbf{area } \boldsymbol{LAKT = \frac{h}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 2 + 4 + 1)},$$

in which merely the coefficients of the successive perpendiculars LA, MB, $\cdots$, TK are written. As in the case of the trapezoidal rule, the greater the number of equal parts into which LT is divided, the more nearly equal will the area thus calculated be to the true area.

If the equation of the curve AK is $y = f(x)$, and $OL = a$, $OT = b$, and LT is divided into n parts, each equal to $\frac{b-a}{n}$, the lengths of the successive ordinates, LA, MB, $\cdots$ TK, are $f(a)$, $f\left(a + \frac{b-a}{n}\right)$, $f\left(a + 2\frac{b-a}{n}\right)$, $\cdots$, $f(b)$. Hence, on calling these successive lengths, $y_0, y_1, y_2, \cdots y_n$,

$$\int_a^b f(x)\,dx = \frac{b-a}{3n}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots \\ + 2y_{n-2} + 4y_{m-1} + y_n).$$

For the sake of computation, this may be put in the form,

$$\int_a^b f(x)\,dx = \frac{2}{3}\cdot\frac{b-a}{n}\left[\frac{1}{2}(y_0+y_n)+2(y_1+y_3+\cdots+y_{n-1})\right.$$
$$\left.+(y_2+y_4+\cdots+y_{n-2})\right].^*$$

Ex. 1. Evaluate $\int_0^{10} x^4\,dx$ by this method, taking $n = 10$.

Here, $y_0, y_1, y_2, \cdots, y_{10}$, are 0, 1, 81, 625, ⋯, 10,000, respectively; and hence, approximately,

$$\int_0^{10} x^4\,dx = \tfrac{20}{30}\{\tfrac{10000}{2}+2(1+81+625+2401+6561)+(16+256+1296+4096)\}$$
$$= 20001\tfrac{1}{3}.$$

The true value of the given integral is 20,000; thus the error is only $1\frac{1}{3}$ in 20,000.

Ex. 2. Show that the value of $\int_2^{12}\log_{10} x\,dx$ calculated by this rule for $n = 10$, is 8.004704 (compare Ex. 3, Art. 85).

A comparison between these two rules is given in the following quotation:† "The increase in accuracy (of the parabolic) over the trapezoidal rule is usually quite notable, unless the number of ordinates become large, in which case they both approximate more and more closely to the true value and to each other. In a

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* If n be the number of equal intervals into which the range $b-a$ is divided, the outside limit of error that the parabolic formula for integration can have, is

$$-\left(\frac{b-a}{2}\right)^5\frac{f^{\text{iv}}(x_r)}{90\,n^4},$$

in which x_r is some value of x between a and b, and $f^{\text{iv}}(x)$ denotes the fourth derivative of $f(x)$. The outside limit of error in the case of the trapezoidal rule is

$$-\frac{(b-a)^3}{12\,n^2}f''(x_r),$$

in which $f''(x)$ denotes the second derivative of $f(x)$. If n is doubled, the limit of error is reduced, therefore, to $\frac{1}{16}$ and $\frac{1}{4}$ of its former amount. (See Boussinesq, *Cours d'Analyse*, t. II. 1, § 262, and Markoff, *Differenzenrechnung*, § 14, pp. 57, 59.)

† This is from an article, entitled, "New Rules for Approximate Integration," in the *Engineering News* (N. Y.), January 18, 1894, by Professor W. F. Durand of Cornell University.

series of trials made by the author upon a number of integrals of various forms for the purpose of testing the relative accuracy of these rules, it was found for cases in which the locus was of single curvature only that the trapezoidal rule required about double the number of sections for equal accuracy with the parabolic rule. Where the locus involves several changes of curvature, as in lumpy and irregular curves, and the number of sections is moderate, one rule is as likely to be right as the other, and both are likely to be considerably in error. For a large number of sections, however, the parabolic rule will show its superiority as above."

87. Durand's rule. From a discussion * on the trapezoidal and parabolic rules, Professor Durand has deduced another rule for which "it seems not unfair to claim substantially the full probable accuracy of the parabolic rule, and practically the simplicity in use of the trapezoidal rule." It is as follows, merely the coefficients of the successive ordinates being written in order from the left:

$$\text{approximate area} = h\left[\tfrac{5}{12} + \tfrac{13}{12} + 1 + 1 + \cdots + 1 + 1 + \tfrac{13}{12} + \tfrac{5}{12}\right];$$

or, approximately,

$$\textbf{area} = \boldsymbol{h}[.4 + 1.1 + 1 + 1 + \cdots + 1 + 1 + 1.1 + .4].$$

The number of intervals may be even or odd.

Ex. 1. Find the value of $\int_0^{60^\circ} \sin\theta\, d\theta$ with 10° intervals.

The circular measure of 10° is .17453. The rule gives for the approximate value of the integral,

$$\int_0^{60^\circ} \sin\theta\, d\theta = .17453\,[.4(\sin 0^\circ + \sin 60^\circ) + 1.1(\sin 10^\circ + \sin 50^\circ) + (\sin 20^\circ + \sin 30^\circ + \sin 40^\circ)] = .5000075.$$

Since the exact value of $\int_0^{60^\circ} \sin\theta\, d\theta$ is $\Big[-\cos\theta\Big]_0^{60^\circ}$, or .5, the difference between the above approximate and the true values of this integral is not more than one part in 66,666.

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* In the article mentioned in the preceding footnote.

Ex. 2. Show that $\int_0^{10} x^2\,dx$ calculated by this rule with unit intervals gives a difference of one part in 3333.

Ex. 3. Show that $\int_2^{12} \log_{10} x$ calculated by this rule with unit intervals is 8.004062. (Compare with Ex. 3, Art. 85, and Ex. 2, Art. 86.)

88. The planimeter. Attention has been drawn to the fact that the value of a definite integral is also the value of a certain plane area, and that, consequently, the measurement of the area is equivalent to the evaluation of the integral. In Arts. 85, 86, 87, rules are given for approximately determining plane areas, and other rules therefor are given in the Appendix.* These areas can be measured exactly by instruments called mechanical integrators or planimeters. A planimeter measures the area of any plane figure by the passage of a tracer round about the perimeter of the figure, the readings being given by a self-recording apparatus. There are several kinds of planimeters, but they all have certain fundamental properties in common. The first planimeter was invented by the Bavarian engineer, J. M. Hermann, in 1814. Amsler's polar planimeter, which was invented by Jacob Amsler when a student at Königsberg in 1854, is the most popular on account of its simplicity and handiness in use. Thousands of them have been made at his works in Schaffhausen.

The Amsler planimeter is shown in Fig. 55. It consists of two bars, (a) the radius bar, and (b) the pole arm, jointed at the point C. The tracing point P, which now coincides with the point B of the figure $ABDE$, is carried round the curve, and the roller m, which partly rolls and partly slips, gives the area of the figure; and by means of the graduated dial h, and the vernier v in connection with the roller m, the result is given correctly in four figures. The sleeve H can be placed in different positions along the pole-arm b, and fixed by a screw s so as to give readings in different required units. A weight at w is placed upon the bar to keep the needle point in its place, but in instruments by some other makers T is a pivot in a much larger weight, which rests on the paper. The accuracy of the reading depends upon the accuracy with which the tracing point follows the curve.*

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* See Note E.

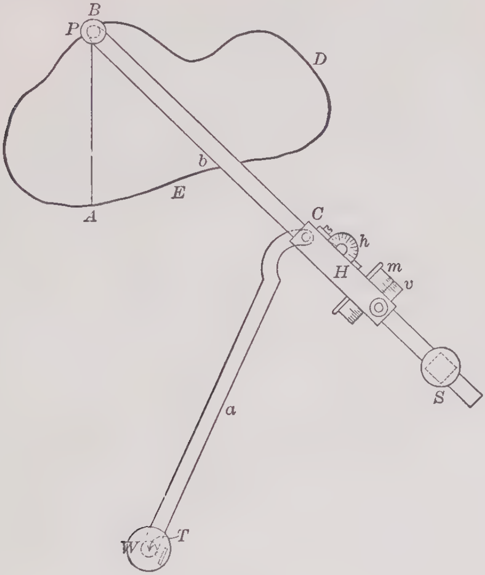

FIG. 55.

Professor O. Henrici's *Report on Planimeters* (Report of the British Association for the Advancement of Science, 1894, pp. 496–523) contains a sketch of the history of planimeters, the geometrical theory of generating areas, descriptions of early planimeters, a discussion on Amsler's planimeter, and a description of some recent planimeters. Professor H. S. Hele Shaw's paper on *Mechanical Integrators* (Proceedings of the Institution of Civil Engineers, Vol. 82, 1885, pp. 75–143) gives an account of the theory and the practical advantages of several varieties of planimeters. The description given above is from this paper. An explanation of the theory of Amsler's planimeter is given by Mr. J. MacFarlane Gray in Carr's *Synopsis of Mathematics*. There is a discussion on planimeters in Professor R. C. Carpenter's *Text-book of Experimental Engineering*, pp. 24–49.

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* For the fundamental theory of the planimeter, see Note F, Appendix.

CHAPTER XII

INTEGRAL CURVES

89. Introduction. A first integral curve was defined in Art. 15. The student is advised to review that article thoroughly before proceeding further. In this chapter the subject of integral curves will be studied more fully, and some of their applications to mechanics will be pointed out. Differentiation under the sign of integration is an important topic in the integral calculus. Only a very special case, however, is necessary in what follows: this case is considered in Art. 90. Arts. 92, 93, 94, contain an exposition of the simpler properties of integral curves and a few examples of their usefulness. Their applications are of especial value to the student of engineering. For the proper understanding of several of them, a better acquaintance with the theorems of mechanics is required than some readers of the calculus may be presumed to have at this stage. Accordingly, a further exposition of the service that may be rendered by these curves is given in the Appendix for purposes of future reference. Articles 94, 95, discuss the practical plotting of integral curves.*

90. Special case of differentiation under the sign of integration. A special case of differentiation under the sign of integration with respect to one of the limits which is also involved in the function under the sign will be considered. Let

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* Arts. 91–95 and the related matter in the Appendix are taken with some slight but no essential change, from an article entitled *Integral Curves*, by Professor W. F. Durand, Principal of the Graduate School of Marine Engineering and Naval Architecture, Cornell University. The article, which appeared in the *Sibley Journal of Engineering*, January, 1897, is practically all reproduced here. This chapter has also had the benefit of Professor Durand's revision.

$$I = \int_0^b (b - x)^n f(x)\, dx. \tag{1}$$

The differential coefficient $\frac{dI}{db}$ will be derived by the fundamental method employed in differentiation, namely, by giving an increment to the variable, in this case b, then finding the corresponding increment in I, and finally obtaining the ratio of these two increments when the increment of b approaches zero.

Suppose that b receives an increment Δb, then from (1)

$$I + \Delta I = \int_0^{b+\Delta b} (b + \Delta b - x)^n f(x)\, dx.$$

Hence

$$\Delta I = \int_0^{b+\Delta b} (b + \Delta b - x)^n f(x)\, dx - \int_0^b (b - x)^n f(x)\, dx;$$

whence, by Art. 7 (*b*),

$$\begin{aligned}\Delta I &= \int_0^b (b + \Delta b - x)^n f(x)\, dx + \int_b^{b+\Delta b} (b + \Delta b - x)^n f(x)\, dx \\ &\quad - \int_0^b (b - x)^n f(x)\, dx = \int_0^b [(b + \Delta b - x)^n - (b - x)^n] f(x)\, dx \\ &\quad + \int_b^{b+\Delta b} (b + \Delta b - x)^n f(x)\, dx.\end{aligned}$$

From this, by Art. 7 (*c*),

$$\begin{aligned}\Delta I &= \int_0^b [(b + \Delta b - x)^n - (b - x)^n] f(x)\, dx \\ &\quad + \Delta b \{b + \Delta b - (b + \theta \cdot \Delta b)\}^n f(b + \theta \cdot \Delta b),\end{aligned}$$

in which $\theta < 1$.

Hence, remarking that Δb is independent of x, and can therefore be put under the integration sign,

$$\frac{\Delta I}{\Delta b} = \int_0^b \frac{[(b + \Delta b - x)^n - (b - x)^n]}{\Delta b} f(x)\, dx + \{(1 - \theta)\,\Delta b\}^n f(b + \theta \cdot \Delta b).$$

Therefore, letting Δb approach zero,

$$\frac{dI}{db} = n\int_0^b (b-x)^{n-1} f(x)\,dx. \tag{2}$$

This result will be required in Art. 93.

91. Integral curves defined. Their analytical relations. A more general definition of an integral curve than that given in Art. 15 will now be introduced. In what follows, a number of curves will be spoken of together. In order to distinguish between them, the system of ordinates, that is, the y's, for each of the several curves will be denoted by a subscript number.

If

$$y = f(x),$$

or, for the sake of distinction,

$$y_0 = f(x) \tag{1}$$

be the equation of a given curve, the curve whose equation is

$$y_1 = \frac{1}{a}\int_0^x y_0\,dx \tag{2}$$

is called a *first integral curve* of the curve whose equation is (1). The latter is called the fundamental curve. Since $\int_0^x y\,dx$ is of the second dimension, and y_1 should be linear, the constant factor $\frac{1}{a}$ is introduced in (2), in which a is a linear quantity and has a magnitude that will make equation (2) convenient for plotting. It may be called a *scale factor*. In the definition in Art. 15 the scale factor was unity.

From (2) on differentiation,

$$\frac{dy_1}{dx} = \frac{1}{a}\,y_0.$$

Hence, as x varies, the slopes of the first integral curve vary as the ordinates of the fundamental; and therefore the former can be represented by the latter, and *vice versa*.

The first integral curve (2) also has a first integral, the latter has a first integral, and so on. These successive integral curves are called the second, third, etc., integral curves of the original or fundamental curve. On using the constant linear quantities, b, c, $\cdots w$, as scale factors for the sake of plotting the curves conveniently, and on distinguishing by different subscripts the ordinates that belong to the various curves, the latter will have the following equations:

Fundamental, $y_0 = f(x)$; (1)

first integral, $$y_1 = \frac{1}{a}\int_0^x y_0\,dx; \tag{2}$$

second integral, $$y_2 = \frac{1}{b}\int_0^x y_1\,dx = \frac{1}{ab}\int_0^x\int_0^x y_0\,dx^2; \tag{3}$$

third integral, $$y_3 = \frac{1}{c}\int_0^x y_2\,dx = \frac{1}{abc}\int_0^x\int_0^x\int_0^x y_0\,dx^3; \tag{4}$$

. . . .

nth integral curve,

$$y_n = \frac{1}{w}\int_0^x y_{n-1}\,dx = \frac{1}{abc\cdots w}\int_0^x\int_0^x\int_0^x\cdots\int_0^x y_0\,dx^n. \tag{5}$$

From equation (2)

$$\frac{dy_1}{dx} = \frac{1}{a}y_0; \tag{6}$$

from (3) $$\frac{dy_2}{dx} = \frac{1}{b}y_1 = \frac{1}{ab}\int_0^x y_0\,dx;$$

and hence, $$\frac{d^2y_2}{dx^2} = \frac{1}{ab}y_0. \tag{7}$$

And in general,

$$\frac{d^ny_n}{dx^n} = \frac{1}{abc\cdots w}y_0. \tag{8}$$

Equation (8) shows that as x varies, the nth derivatives of the nth integral curve vary as the ordinates of the fundamental curve; and therefore, the former can be represented by the latter.

92. Simple geometrical relations of integral curves. In Fig. 56 RP, OA, OB, OC, represent the fundamental, and the first, second, and third integral curves respectively, whose equations are (1), (2), (3), (4), of Art. 91.

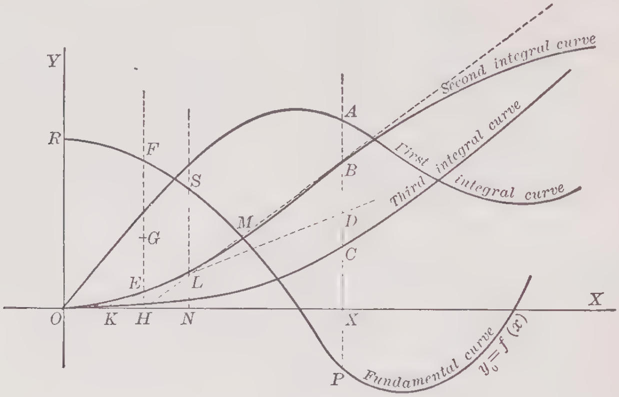

Fig. 56.

(a) As x increases, and so long as the fundamental curve RP lies above the x-axis, the ordinates of the first integral OA will increase, and the tangent to OA will make a positive angle with the x-axis; when RP lies below the x-axis the tangent to OA makes a negative angle with the x-axis; when RP crosses the x-axis, the tangent to OA is parallel to the x-axis. These properties follow from equations (2) and (6), Art. 91.

(b) At points for which the ordinate of the first integral curve is a maximum or a minimum,

$$\frac{dy_1}{dx} = 0;$$

and there also, by (6), $y_0 = 0.$

Hence, to a zero value of the ordinate of the fundamental there corresponds a maximum or a minimum value of an ordinate of the first integral curve.

(c) At points where an ordinate of the fundamental is a maximum or a minimum,

$$\frac{dy_0}{dx}=0;$$

and at points where the first integral curve has a point of inflexion,

$$\frac{d^2y_1}{dx^2}=0.$$

Differentiation of the members of equation (6) shows that

$$\frac{d^2y_1}{dx^2}=0 \text{ when } \frac{dy_0}{dx}=0.$$

Hence, to a point on the fundamental at which there is a maximum or a minimum value of the ordinate, there corresponds a point of inflexion on the first integral curve.

93. Simple mechanical relations and applications of integral curves. Successive moments of an area about a line. If each infinitesimal portion of a plane area be multiplied by its distance from a given line, the sum of all these products is called the *moment of first degree* of the area about the line. If each of the infinitesimal portions of the area be multiplied by the square of its distance from the given line, the sum of all the products is called the *moment of second degree* of the area with respect to the line. The latter is the moment of inertia of the area about the line, examples of which were shown in Art. 80. The moment of first degree is usually called the statical moment. In general, if each infinitesimal portion of an area be multiplied by the nth power of its distance from a given line, the sum of all these products is called the *moment of the nth degree* of the area about the line. For the sake of brevity, this may be called the nth moment.

Thus (Fig. 56), lay off $OX=x_1$, and erect the ordinate AP at X, and consider $\int_0^{x_1} y_0\,dx$, the area $ORPX$, between the fundamental, the axes, and the ordinate AP. If the successive moments of this area be taken about the ordinate AP for which $x = x_1$, and these moments be denoted by $M_1, M_2, \cdots, M_n$, in order, then

the first moment, $$M_1 = \int_0^{x_1}(x_1 - x)y\,dx; \tag{1}$$

the second moment, $$M_2 = \int_0^{x_1}(x_1 - x)^2 y\,dx; \tag{2}$$

. . . .

the nth moment, $$M_n = \int_0^{x_1}(x_1 - x)^n y\,dx. \tag{3}$$

In this notation, $$M_0 = \int_0^{x_1}(x_1 - x)^0 y\,dx,$$

$$= \int_0^{x_1} y\,dx, \text{ the area.} \tag{4}$$

(a) Differentiation of (3) with respect to x_1 will give by equation (2), Art. 90,

$$\frac{dM_n}{dx_1} = n\int_0^{x_1}(x_1 - x)^{n-1}y\,dx;$$

that is, $$\frac{dM_n}{dx_1} = nM_{n-1}. \tag{5}$$

Hence, $$M_n = n\int_0^{x_1} M_{n-1}\,dx_1.$$

Since dx_1 is an infinitesimal distance along the x-axis, it can be written dx, and hence

$$M_n = n\int_0^{x_1} M_{n-1}\,dx. \tag{6}$$

By successive application of (6) there will finally be obtained,

$$M_n = n!\int_0^{x_1}\int_0^{x_1}\cdots\int_0^{x_1} M_0\,dx^n. \tag{7}$$

That is, the nth moment of the area $ORPX$ about an ordinate distant x_1 from the origin is equal to factorial n times the nth integral of the moment of degree zero for the same area. On substituting for M_0 in (7) its value from (4), there is obtained,

$$M_n = n!\int_0^{x_1}\int^{x_1}\cdots\int_0^{x_1}\int_0^{x_1} y\,(dx)^{n+1}. \qquad (8)$$

Hence, the value of the nth moment of the area of $y = f(x)$ above described about an ordinate distant x_1 from the origin, is equal to factorial n times the ordinate of the $(n+1)$th integral curve at $x = x_1$; and, therefore, the nth moment may be represented by this ordinate. On using Y_{n+1} to denote the ordinate of the $(n+1)$th integral curve at $x = x_1$, this may be expressed by

$$M_n = n!\ Y_{n+1}.$$

In particular, the statical moment (1) is represented by the corresponding ordinate of the second integral curve, and the moment of inertia (2) by twice the corresponding ordinate of the third integral curve. Thus in Fig. 56,

the area $ORPX$ is represented by AX;
its statical moment about AP is represented by BX;
and its moment of inertia about AP by $2\,CX$.

Suppose that the scale factors used in plotting the three integral curves, each from the one of next lower order, are a, b, c, respectively, as indicated in equations (2), (3), (4), Art. 91. Then,

$$\text{area } ORPX = a \cdot AX;$$

the statical moment of $ORPX$ about $AP = ab \cdot BX$;
the moment of inertia of $ORPX$ about $AP = 2\,abc \cdot CX$.

(b) If G is the center of mass (or center of gravity) of $ORPX$, its distance HX from AP, by Art. 79, is determined thus:

$$HX = \frac{\int_0^{x_1}(x_1 - x)\,y\,dx}{\int_0^{x_1} y\,dx} = \frac{M_1}{M_0} = \frac{aby_2}{ay_1} = b\,\frac{BX}{AX}.$$

(c) If k is the radius of gyration of the area $ORPX$ about AP, then by (Art. 80),

$$k^2 = \frac{\text{Moment of Inertia about } AP}{\text{Area}} = \frac{2\,abcy_3}{ay_1} = 2\,bc\frac{CX}{AX}.$$

For further applications to mechanics, and some general remarks on the use of these curves in engineering problems, see Appendix. The reader is recommended to glance at the latter remarks now.

94. Practical determination of an integral curve from its fundamental curve. The integraph. Suppose that the equation of the fundamental curve is $y = f(x)$. The ordinates of the first integral curve that correspond to successive values $x_1, x_2, \cdots, x_n$, of the abscissas are

$$\frac{1}{a}\int_0^{x_1} y\,dx,\ \frac{1}{a}\int_0^{x_2} y\,dx,\ \cdots,\ \frac{1}{a}\int_0^{x_n} y\,dx,$$

respectively. These may be determined by the ordinary rules for integration when the functions under the sign of integration are integrable. If the latter condition does not hold, recourse can be had to some of the various methods of mechanical and approximate integration described in Arts. 85–88. It will be necessary to do this also, when the fundamental has been plotted merely from a knowledge of the ordinates that correspond to particular abscissas, the equation of the curve being unknown.

For example, in Fig. 57, the area of each successive section between the ordinates of the fundamental may be found with a planimeter, and the ordinates of the integral curve, which is shown by the dotted line, may be found by successive additions. As an instrumental check, it is well from time to time to go around the entire area between the y-axis and the ordinate in question, and compare the result with the total area summed to that point. Numerical means of integration may also be employed. The trapezoidal rule and the parabolic rule can be readily used for finding successive increments of area in the case of the fundamental, and hence for finding successive increments of the ordinates of the first integral curve.

In whatever way the integral curve may be derived from the fundamental, it is well, after plotting, to compare the two and note the fulfillment of the simple geometrical relations, (a), (b), (c), of Art. 92. Thus, one should look for a maximum or a minimum ordinate in the integral corresponding to every zero ordinate in the fundamental, and for a point of inflexion in the integral for each maximum or minimum in the fundamental. The tangent of the integral varies with the ordinate of the fundamental, and hence, the slope of the integral should increase or decrease when the ordinate of the fundamental increases or decreases. These relations may be noted in the curves in Figs. 56, 57.

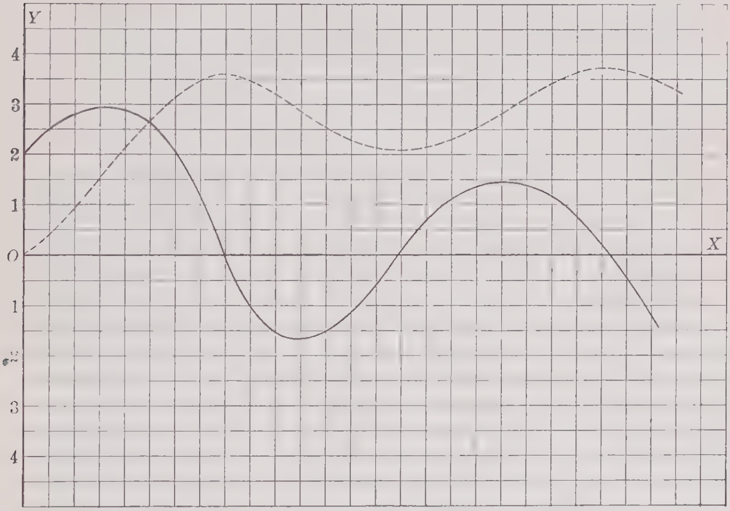

FIG. 57.

The *integraph* is an instrument that is used for drawing the first integral curve from its fundamental. The theory of it is given in the Appendix (Note G). It may be used also for determining the area between a curve and the x-axis. For the integral curve can be drawn with the integraph, and the ordinate corresponding to the area can be measured. Since the length of the ordinate represents the area, the latter can be found immediately on making allowance for the scale-factor

95. The determination of scales. In order to have the various curves convenient for plotting, it is usually necessary to employ different scales for the ordinates. If numerical integration is used, the value of the area of the fundamental will be found directly, and the scale may be correspondingly selected so that the curve will be kept within the desired limits as to size. If the planimeter is used, the result will be given in square inches or other area units, and must be converted into the value desired by the use of a scale factor. Suppose the fundamental plotted as follows:

horizontally 1 unit of length $= p$ units of abscissa,
vertically 1 unit of length $= q$ units of ordinates.

Then 1 unit of area on the diagram will represent pq units of the integrated function, and the area found must be multiplied by this factor in order to reduce it to the value of the integral desired. The scale factors a, b, c, etc., may then be chosen as before.

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* See Note G.

CHAPTER XIII

ORDINARY DIFFERENTIAL EQUATIONS

96. Differential equation, order, degree. A few differential equations which frequently appear in practical work will now be discussed very briefly.*

A differential equation is an equation that involves differentials or differential coefficients. Ordinary differential equations are those which contain only one independent variable. For example,

$$dy = \cos x\, dx, \tag{1}$$

$$\frac{d^2y}{dx^2} = 0, \tag{2}$$

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3}}{\frac{d^2y}{dx^2}}, \tag{3}$$

$$y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}, \tag{4}$$

$$(y + c)^2\frac{dx}{dz} + z\frac{dy}{dz} - (y + a) = 0, \tag{5}$$

are ordinary differential equations.

The order of a differential equation is the order of the highest derivative that appears in it. The degree of a differential equation is the degree of the highest derivative when the equation is free from radicals and fractions. Of the examples above, (1) is of the first order and first degree, (2) is of the second order and first degree, (3) is of the second order and second degree, (4) is of the first order and second degree, (5) is of the first order and first degree. Differential equations of a very simple kind have already been considered.

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* For fuller explanations than are given here, reference may be made to *Introductory Course in Differential Equations*, by D. A. Murray. (Longmans, Green, & Co.)

97. Constants of integration. General and particular solutions. Derivation of a differential equation. If a relation between the variables together with the derivatives obtained therefrom satisfies a differential equation, the relation is called a *solution* or *integral* of the differential equation. For example,

$$y = m \sin x, \tag{1}$$

$$y = n \cos x, \tag{2}$$

$$y = A \cos x + B \sin x, \tag{3}$$

$$y = c \sin (x + \alpha), \tag{4}$$

in which m, n, A, B, c, α, are arbitrary constants, are all solutions of the equation

$$\frac{d^2y}{dx^2} = 0. \tag{5}$$

This may be verified by substitution. It will be observed that (5) does not contain m, n, A, B, c, or α. The solutions of the differential equations of the first order which have appeared in the former part of this book contain one constant of integration; those of the second order contain two constants. Examples have been given in Arts. 8, 9, 12, 59, etc.

Solutions (1) and (2) above contain one arbitrary constant, and solutions (3) and (4) each contain two. The question is suggested: *How many* arbitrary constants should the most general solution of a differential equation contain? The answer can in part be inferred from a consideration of one of the ways in which a differential equation may arise, namely, by the elimination of constants. On comparing (3) and (5) it is seen that (5) must have been derived from (3) by the elimination of the two constants A and B. In order to eliminate two quantities, three equations are necessary. One of these is given, and the others can be obtained by successive differentiation. Thus,

$$y = A \cos x + B \sin x,$$

$$\frac{dy}{dx} = -A \sin x + B \cos x,$$

$$\frac{d^2y}{dx^2} = -A \cos x - B \sin x;$$

whence,

$$\frac{d^2y}{dx^2} + y = 0.$$

In order to eliminate three constants from a given equation, four equations are required. Of these, one is given and the remaining three can be obtained only by successive differentiation. The third differentiation will introduce a differential coefficient of the third order, which accordingly will appear in the differential equation that is formed by the elimination of the three constants. In general, if an integral relation contains n arbitrary constants, these constants can be eliminated by means of $n+1$ equations. The latter consist of the given equation and n relations obtained by n successive differentiations. The nth differentiation introduces a differential coefficient of the nth order, which will accordingly appear in the differential equation that arises on the elimination of the constants. The solution of an equation of the nth order cannot contain more than n constants; for if it had $n+1$, their elimination would lead to the equation of the $n+1$th order.*

The solution that contains a number of arbitrary constants equal to the order of the equation is called the *general solution* or the *complete integral.* Solutions obtained therefrom by giving particular values to the constants are called *particular solutions.* For example, (3) and (4) are general solutions of (5), and

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* For a proof that the general solution of a differential equation contains exactly n arbitrary constants, see *Introductory Course in Differential Equations*, Art. 3 and Note C, Appendix.

$$y = 2\cos x + 3\sin x,\ y = 5\cos x - \sin x,\ y = 2\sin(x+\pi),$$
$$y = 3\sin\left(x - \frac{\pi}{6}\right),$$

are particular solutions.

Ex. **1.** Eliminate the arbitrary constants m and c from

$$(1)\quad y = mx + c.$$

Differentiating twice, $$(2)\quad \frac{dy}{dx} = m,$$

$$(3)\quad \frac{d^2y}{dx^2} = 0.$$

These equations may be interpreted geometrically. If m, c, are both arbitrary, (1) is the equation of any straight line; and, therefore, (3) is the differential equation of all straight lines. If c is arbitrary and m has a definite value, (1) is the equation of any line that has the slope m, and, accordingly, (2) is the differential equation of all the straight lines that have the slope m.

Ex. **2.** Find the differential equation of all circles of radius r.

The equation of any circle of radius r is

$$(x-a)^2 + (y-b)^2 = r^2,$$

in which a, b, the coördinates of the center, are arbitrary. The elimination of a and b gives

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = r\frac{d^2y}{dx^2},$$

the equation required.

Ex. **3.** If $y = Ax^2 + B$, prove that $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$.

Ex. **4.** Eliminate c from $y = cx + c - c^3$.

Ex. **5.** Form the differential equation of which $e^{2y} + 2cxe^y + c^2 = 0$ is the complete integral.

Ex. **6.** Eliminate the constants from $y = ax + bx^2$.

Ex. **7.** Form the differential equation which has $y = a\cos(mx + b)$ for its complete integral, a and b being arbitrary constants.

SECTION I. EQUATIONS OF THE FIRST ORDER AND THE FIRST DEGREE.

98. Equations in which the variables are easily separable.

If an equation is in the form

$$f_1(x)\,dx + f_2(y)\,dy = 0,$$

its solution, obtainable at once by integration, is

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c.$$

Some equations can easily be put in this form.

Ex. 1. Solve $(1-x)dy - (1+y)dx = 0$.

This may be written, $$\frac{dy}{1+y} - \frac{dx}{1-x} = 0.$$

This step is called "separation of the variables."

Integrating, $$\log(1+y) + \log(1-x) = c_1,$$

or, $$(1+y)(1-x) = e^{c_1} = c.$$

Ex. 2. Solve $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0.$

Ex. 3. Solve $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0.$

99. Equations homogeneous in x and y. If an equation is homogeneous in x and y, the substitution

$$y = vx$$

will give a differential equation in v and x in which the variables are easily separable.

Ex. 1. Solve $(x^2+y^2)dx - 2xy\,dy = 0$.

Rearranging, $$(1)\quad \frac{dy}{dx} = \frac{x^2+y^2}{2xy}.$$

On putting $$y = vx,$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitution of these values in (1) gives

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}.$$

Separating the variables, $\frac{dx}{x} - \frac{2v\,dv}{1 - v^2} = 0.$

Integrating, $\log x(1 - v^2) = \log c;$

that is, $\log x\left(1 - \frac{y^2}{x^2}\right) = \log c;$

whence, $x^2 - y^2 = cx.$

Ex. 2. Solve $y^2\,dx + (xy + x^2)\,dy = 0.$

Ex. 3. Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0.$

Ex. 4. Show that the non-homogeneous equation of the first degree in x and y

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

is made homogeneous, and therefore integrable, by the substitution

$$x = x_1 + h, \quad y = y_1 + k,$$

h, k being solutions of

$$ah + bk + c = 0,$$

$$a'h + b'k + c' = 0.$$

100. Exact differential equations. An exact differential equation is one that is formed by equating an exact differential to zero. It follows from Art. 24 that

$$M\,dx + N\,dy = 0$$

is an exact differential equation if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Ex. 1. Solve $(a^2 - 2xy - y^2)\,dx - (x + y)^2\,dy = 0.$

Ex. 2. Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$

Ex. 3. Solve $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})\,dy$
$+ (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx = 0.$

101. Equations made exact by means of integrating factors. The differential equation

$$y\,dx - x\,dy = 0$$

is not exact. Multiplication by $\frac{1}{y^2}$ gives

$$\frac{y\,dx - x\,dy}{y^2} = 0.$$

This is exact, and its solution is $\frac{x}{y} = c$, or $x = cy$.

When multiplied by $\frac{1}{xy}$, the first equation becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0,$$

which is also exact. The solution is

$$\log x - \log y = \log c,$$

whence,
$$\frac{x}{y} = c, \text{ or } x = cy.$$

Another factor that will make the given equation exact is $\frac{1}{x^2}$. Any factor such as $\frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2}$ employed above, which changes an equation into an exact differential equation, is called an *integrating factor*. It can be shown that the number of integrating factors is infinite. There are several rules for finding integrating factors. In the following examples, the necessary integrating factor can be found by inspection.

Ex. 1. Solve $y\,dx - x\,dy + \log x\,dx = 0$.

Here, $\log x\,dx$ is integrable, but a factor is needed for $y\,dx - x\,dy$. Obviously $\frac{1}{x^2}$ is the factor to be employed, as it will not affect the third term injuriously from the point of view of integration. On multiplication by $\frac{1}{x^2}$ the given equation becomes

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\log x}{x^2} = 0.$$

The solution of this equation reduces to

$$cx + y - \log x - 1 = 0.$$

Ex. 2. $a(x\,dy + 2\,y\,dx) = xy\,dy.$

Ex. 3. $(x^2 + y^2 + 1)\,dx - 2\,xy\,dy = 0.$

Ex. 4. $(x^3e^x - 2\,my^2)\,dx + 2\,m\,xy\,dy = 0.$

102. Linear equations. If the dependent variable and its derivative appear only in the first degree in a differential equation, the latter is said to be linear. The form of the linear equation of the first order is

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

in which P and Q are functions of x, or constants. The linear equation occurs very frequently. The solution of

$$\frac{dy}{dx} + Py = 0,$$

that is, of

$$\frac{dy}{y} = -P\,dx,$$

is

$$y = ce^{-\int Pdx}, \text{ or } ye^{\int Pdx} = c.$$

On differentiation the latter form gives

$$e^{\int Pdx}(dy + Py\,dx) = 0,$$

which shows that $e^{\int Pdx}$ is an integrating factor of (1).

Multiplication of (1) by this factor gives

$$e^{\int Pdx}(dy + Py\,dx) = e^{\int Pdx}Q\,dx;$$

and this, on integration, gives

$$ye^{\int Pdx} = \int e^{\int Pdx}Q\,dx + c,$$

or

$$y = e^{-\int Pdx}\left\{\int e^{\int Pdx}Q\,dx + c\right\}. \tag{2}$$

The latter can be used as a formula for obtaining the value of y in a linear equation of the form (1).

Ex. 1. Solve $x\frac{dy}{dx} - y = x^3$. .

This is linear since y and $\frac{dy}{dx}$ appear only in the first degree. On putting it in the ordinary form (1), it becomes

$$\frac{dy}{dx} - \frac{y}{x} = x^2.$$

Here $P = -\frac{1}{x}$, and hence, the integrating factor $e^{\int P dx} = e^{-\int \frac{dx}{x}} = e^{\log \frac{1}{x}} = \frac{1}{x}$.

By using this factor, and adopting the differential form, the equation is changed into

$$\frac{1}{x} dy - \frac{1}{x^2} y\, dx = x\, dx.$$

Integrating, $\frac{y}{x} = \frac{1}{2}x^2 + c$, or $y = \frac{1}{2}x^3 + cx$.

Ex. 2. Solve $\frac{dy}{dx} + y = e^{-x}$.

Ex. 3. Solve $\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1$.

Ex. 4. Solve $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}$.

Ex. 5. Solve $\frac{dy}{dx} + \frac{n}{x}y = \frac{a}{x^n}$.

103. Equations reducible to the linear form. Sometimes equations that are not linear can be reduced to the linear form. One type of such equations is

$$\frac{dy}{dx} + Py = Qy^n,$$

in which P, Q, are functions of x, and n is any constant. Division by y^n and multiplication by $(-n+1)$ gives

$$(-n+1)\, y^{-n}\frac{dy}{dx} + (-n+1)\, Py^{-n+1} = (-n+1)\, Q.$$

On substituting v for y^{-n+1}, this reduces to

$$\frac{dv}{dx}+(1-n)\,Pv=(1-n)\,Q,$$

which is linear in v.

Ex. 1. Solve $\frac{dy}{dx}+\frac{1}{x}y=x^2y^6$.

Division by y^6 gives $$y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2.$$

On putting v for y^{-5}, this takes the linear form

$$\frac{dv}{dx}-\frac{5}{x}v=-5x^2.$$

The solution is $$v=y^{-5}=cx^5+\tfrac{5}{2}x^3.$$

Ex. 2. Solve $\frac{dy}{dx}+\frac{2}{x}y=3x^2y^{\frac{4}{3}}$.

Ex. 3. Solve $\frac{dy}{dx}+\frac{xy}{1-x^2}=xy^{\frac{1}{2}}$.

Ex. 4. Solve $3\frac{dy}{dx}+\frac{2}{x+1}y=\frac{x^3}{y^2}$.

Ex. 5. Show that the equation

$$f'(y)\frac{dy}{dx}+Pf(y)=Q,$$

in which P, Q, are functions of x, becomes linear on the substitution of v for $f(y)$.

SECTION II. EQUATIONS OF THE FIRST ORDER BUT NOT OF THE FIRST DEGREE.

104. Equations that can be resolved into component equations of the first degree. In what follows, $\frac{dy}{dx}$ will be denoted by p. The type of the equation of the first order and nth degree is

$$f(x, y, p)=0,$$

which on expansion becomes

$$p^n+P_1p^{n-1}+P_2p^{n-2}+\cdots+P_{n-1}p+P_ny=0. \tag{1}$$

Suppose that the first member of (1) can be resolved into rational factors of the first degree so that (1) takes the form,

$$(p - R_1)(p - R_2) \cdots (p - R_n) = 0. \tag{2}$$

Equation (1) is satisfied by any values of y that will make a factor of the first member of (2) equal to zero. Therefore, in order to obtain the solutions of (1), equate each of the factors in (2) to zero, and find the integrals of the n equations thus formed. Suppose that the solutions derived from (2) are

$$f_1(x, y, c_1) = 0, \quad f_2(x, y, c_2) = 0, \quad \cdots, \quad f_n(x, y, c_n) = 0,$$

in which $c_1, c_2, \cdots, c_n$, are arbitrary constants of integration. These solutions are just as general if $c_1 = c_2 = \cdots = c_n$, since each of the c's can take any one of an infinite number of values. The solutions will then be written

$$f_1(x, y, c) = 0, \quad f_2(x, y, c) = 0, \quad \cdots, \quad f_n(x, y, c) = 0,$$

or simply, $\quad f_1(x, y, c) f_2(x, y, c) \cdots f_n(x, y, c) = 0.$

Ex. 1. Solve $\left(\frac{dy}{dx}\right)^2 + (x + y)\frac{dy}{dx} + xy = 0.$

This equation can be written $\quad (p + y)(p + x) = 0.$

The component equations are $\quad p + y = 0, \quad p + x = 0,$

of which the solutions are $\quad \log y + x + c = 0, \quad 2y + x^2 + 2c = 0.$

The combined solution is $\quad (\log y + x + c)(2y + x^2 + 2c) = 0.$

Ex. 2. Solve $\left(\frac{dy}{dx}\right)^3 = ax^4.$ **Ex. 3.** Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$

105. Equations solvable for *y*. When equation (1), Art. 104, cannot be resolved into component equations, it may be solvable for y. In this case,

$$f(x, y, p) = 0$$

can be put in the form $\quad y = F(x, p).$

Differentiation with respect to x gives

$$p = \phi\left(x, p, \frac{dp}{dx}\right),$$

which is an equation in two variables x and p. From this it may be possible to deduce a relation

$$\psi(x, p, c) = 0.$$

The elimination of p between the latter and the original equation gives a relation that involves x, y, c. This is the solution required. If the elimination of p is not easily practicable, the values of x and y in terms of p as a parameter can be found, and these together will constitute the solution.

Ex. 1. Solve $x - yp = ap^2$.

Here
$$y = \frac{x - ap^2}{p}.$$

Differentiating with respect to x, and clearing of fractions,

$$(ap^2 + x)\frac{dp}{dx} = p(1 - p^2).$$

This can be put in the linear form

$$\frac{dx}{dp} - \frac{1}{p(1-p^2)}x = \frac{ap}{1-p^2}.$$

Solving,
$$x = \frac{p}{\sqrt{1-p^2}}(c + a\sin^{-1}p).$$

Substituting in the value of y above,

$$y = -ap + \frac{1}{\sqrt{1-p^2}}(c + a\sin^{-1}p).$$

Ex. 2. Solve $4y = x^2 + p^2$.

Ex. 3. Solve $y = 2p + 3p^2$.

106. Equations solvable for x. In this case $f(x, y, p)$ can be put in the form

$$x = F(y, p).$$

Differentiation with respect to y gives

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right),$$

from which a relation between p and y may sometimes be obtained, say,
$$f(y, p, c) = 0.$$

Between this and the given equation p may be eliminated, or x and y may be expressed in terms of p as in the last article.

Ex. 1. Solve $x = y + a \log p$.

Ex. 2. Solve $x(1 + p^2) = 1$.

107. Clairaut's equation. Any differential equation of the first order which is in the first degree in x and y comes under the cases discussed in Arts. 105, 106. An important equation of this kind is that of Clairaut.* It has the form

$$y = px + f(p). \tag{1}$$

Differentiation with respect to x gives

$$p = p + \{x + f'(p)\}\frac{dp}{dx}.$$

From this,

$$x + f'(p) = 0, \tag{2}$$

or,

$$\frac{dp}{dx} = 0. \tag{3}$$

From the latter equation it follows that $p = c$. Substitution of this value in the given equation shows that

$$y = cx + f(c) \tag{4}$$

is the general solution. See *Introductory course in differential equations*, Art. 28, for remark on equation (2). Some equations are reducible to Clairaut's form, for instance, Ex. 2 below.

Solution (4) represents a family of straight lines. The envelope of this family of lines will also satisfy the differential equation, since x, y, p, at any point on the envelope is identical with the x, y, p of some point on one of the tangent lines of which (4) is the equation. The equation of the envelope of (4) is called the *singular solution* of (1). Singular solutions are discussed in Chapter IV. of the work referred to above.

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* Alexis Claude Clairaut (1713–1765) was the first person who had the idea of aiding the integration of differential equations by differentiating them. He applied it to the equation that now bears his name, and published the method in 1734.

Ex. 1. Solve $y = px + a\sqrt{1 + p^2}$.

The general solution, obtained by the substitution of c for p, is

$$y = cx + a\sqrt{1 + c^2},$$

which represents a family of lines. The envelope of this family is the circle

$$x^2 + y^2 = a^2.$$

The latter equation is the singular solution.

Ex. 2. $x^2(y - px) = yp^2$.

On putting $x^2 = u$, $y^2 = v$, the equation becomes

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2,$$

which is Clairaut's form. The solution is

$$v = cu + c^2,$$

that is, $$y^2 = cx^2 + c^2.$$

Ex. 3. $y = px + \sin^{-1} p$.

Ex. 4. $py = p^2x + m$.

Ex. 5. $xy^2 = pyx^2 + x + py$.

108. Geometrical applications. Orthogonal trajectories. A curve is often defined by some property whose expression takes the form of a differential equation. In the examples given below the differential equations of the curves are of a less simple character than those which appeared in similar problems in Arts. 8, 12, 32.

Problems that relate to orthogonal trajectories are of considerable importance. Suppose that there is a singly infinite system of curves

$$f(x, y, a) = 0, \tag{1}$$

in which a is a variable parameter. The curves which cut all the curves of the given system at right angles are called orthogonal *trajectories* of the system. The elimination of a from (1) gives an equation of the form

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0, \tag{2}$$

the differential equation of the given family of curves. If two curves cut at right angles, and if ϕ_1, ϕ_2, be the angles which the tangents at the intersection make with the axis of x, then

$$\phi_1 = \phi_2 \pm \frac{\pi}{2},$$

and therefore, $\tan\phi_1 = -\cot\phi_2.$

Hence, $\frac{dy}{dx}$ for one curve is the same as $-\frac{1}{\frac{dy}{dx}}$ for the other. Therefore, the differential equation of the system of orthogonal trajectories is obtained by substituting

$$-\frac{dx}{dy} \text{ for } \frac{dy}{dx}$$

in equation (2). This gives

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0.$$

Integration will give the equation in the ordinary form.

Suppose that $f(r, \theta, c) = 0$

is the equation of the given family in polar coördinates, and that

$$\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \tag{3}$$

is the corresponding differential equation obtained by the elimination of the arbitrary constant c. Let ψ_1, ψ_2, denote the angles which the tangents to one of the original curves and a trajectory at their point of intersection make with the radius vector to the point. Then

$$\tan\psi_1 = -\cot\psi_2.$$

Now $\tan\psi_1 = r\frac{d\theta}{dr}$. Hence the differential equation of the required family of trajectories is obtained by substituting

$$-\frac{1}{r}\frac{dr}{d\theta} \text{ for } r\frac{d\theta}{dr},$$

that is, $$-r^2\frac{d\theta}{dr} \text{ for } \frac{dr}{d\theta}$$

in (3). This gives $$\phi\left(r, \theta, -r^2\frac{d\theta}{dr}\right)$$

as the differential equation of the orthogonal system.

Ex. **1**. Find the orthogonal system of the family of parabolas

$$y^2 = 4\,ax.$$

Differentiating, $$y\frac{dy}{dx} = 2\,a,$$

and eliminating a, $$y = 2\,x\frac{dy}{dx}.$$

This is the differential equation of the given family. Substitution of

$$-\frac{dx}{dy} \text{ for } \frac{dy}{dx}$$

gives $$y = -2\,x\frac{dx}{dy},$$

the differential equation of the family of trajectories. Integration gives

$$y^2 + 2\,x^2 = c^2,$$

the equation of a family of ellipses whose foci are on the y-axis, and whose centers are at the origin.

Ex. **2**. Find the orthogonal trajectories of the cardioids

$$r = a(1 - \cos\theta).$$

Differentiating, $$\frac{dr}{d\theta} = a\sin\theta.$$

Elimination of a gives $$\frac{dr}{d\theta} = r\cot\frac{\theta}{2},$$

the differential equation of the given family of curves. Therefore, the equation of the system of trajectories is

$$-r\frac{d\theta}{dr} = \cot\frac{\theta}{2}.$$

Integration gives $$r = c\,(1 + \cos\theta),$$

another system of cardioids.

Ex. 3. Find the curve in which the perpendicular upon a tangent from the foot of the ordinate of the point of contact is constant and equal to a, determining the constant of integration in such a manner that the curve shall cut the axis of y at right angles.

Ex. 4. Find the curve whose tangents cut off intercepts from the axes the sum of which is constant.

Ex. 5. Find the curve in which the perpendicular from the origin upon any tangent is of constant length a.

Ex. 6. Find the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact.

Ex. 7. Find the orthogonal trajectories of the straight lines $y = cx$.

Ex. 8. Find the curves orthogonal to the circles that touch the y-axis at the origin.

Ex. 9. Find the orthogonal trajectories of the family of hyperbolas

$$xy = k^2.$$

Ex. 10. Find the orthogonal trajectories of the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1,$$

in which λ is arbitrary.

Ex. 11. Show that the system of confocal and coaxial parabolas $y^2 = 4\,a(x + a)$ is self-orthogonal.

Ex. 12. Find the orthogonal trajectories of the system of circles

$$r = c \cos \theta,$$

which pass through the origin and have their centers on the initial line.

Ex. 13. Find the orthogonal trajectories of the system of curves

$$r^n \sin n\theta = a^n.$$

Ex. 14. Find the equation of the system of orthogonal trajectories of the family of confocal and coaxial parabolas $r = \dfrac{2\,a}{1 + \cos \theta}$.

Ex. 15. Determine the orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$; and therefrom find the orthogonal trajectories of the series of lemniscates $r^2 = a^2 \cos 2\,\theta$.

SECTION III. EQUATIONS OF AN ORDER HIGHER THAN THE FIRST.

109. Equations of the form $\frac{d^n y}{dx^n} = f(x)$. The solutions of equations of this type can be obtained by n successive integrations. Examples have already been seen in Art. 59.

Ex. 1. Solve $\frac{d^4 y}{dx^4} = x^2 - 2\cos x + 3.$

Integrating, $\frac{d^3 y}{dx^3} = \frac{1}{3}x^3 - 2\sin x + 3x + c_1.$

Integrating, $\frac{d^2 y}{dx^2} = \frac{1}{12}x^4 + 2\cos x + \frac{3}{2}x^2 + c_1 x + c_2.$

Integrating, $\frac{dy}{dx} = \frac{1}{60}x^5 + 2\sin x + \frac{x^3}{2} + \frac{c_1 x^2}{2} + c_2 x + c_3.$

Integrating, $y = \frac{1}{360}x^6 - 2\cos x + \frac{1}{8}x^4 + k_1 x^3 + k_2 x^2 + c_3 x + c_4.$

Ex. 2. Solve $\frac{d^2 x}{dt^2} = f \sin nt.$

Ex. 3. Solve $\frac{d^2 x}{dt^2} = g.$

Ex. 4. Solve $B\frac{d^2 y}{dx^2} - W(l - x) = 0$, subject to the condition that $y = 0$ and $\frac{dy}{dx} = 0$ for $x = 0$.

Ex. 5. Solve $\frac{d^3 y}{dx^3} = xe^x.$

Ex. 6. Solve $\frac{d^n y}{dx^n} = x^m.$

110. Equations of the form $\frac{d^2 y}{dx^2} = f(y)$. Multiplication of both members of this equation by $2\frac{dy}{dx}$ gives

$$2\frac{dy}{dx}\frac{d^2 y}{dx^2} = 2f(y)\frac{dy}{dx}.$$

Integrating, $$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c_1,$$

whence, $$\frac{dy}{\left\{2\int f(y)\,dy+c_1\right\}^{\frac{1}{2}}}=dx.$$

Therefore, $$\int\frac{dy}{\left\{2\int f(y)\,dy+c_1\right\}^{\frac{1}{2}}}=x+c_2.$$

Ex. 1. Solve $$\frac{d^2y}{dx^2}+a^2y=0.$$

Multiplying by $2\frac{dy}{dx}$, $$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=-2a^2y\frac{dy}{dx};$$

and integrating, $$\left(\frac{dy}{dx}\right)^2=-a^2y^2+c,$$

or, putting $a^2c_1^2$ for c, $$=a^2(c_1^2-y^2).$$

Separating the variables, $$\frac{dy}{\sqrt{c_1^2-y^2}}=adx,$$

and integrating, $$\sin^{-1}\frac{y}{c_1}=ax+c_2.$$

Therefore, $$y=c_1\sin(ax+c_2).$$

This solution may also be written $y=A\sin ax+B\cos ax$.

Ex. 2. Solve $\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}$, determining the constants, so that $\frac{dx}{dt}=0$ when $x=a$, and that $x=a$ when $t=0$.

Ex. 3. Solve $\frac{d^2y}{dx^2}-a^2y=0$.

111. Equations in which y appears in only two derivatives whose orders differ by unity. The typical form of these equations is

$$f\left(\frac{d^ny}{dx^n},\ \frac{d^{n-1}y}{dx^{n-1}},\ x\right)=0.$$

If p be substituted for $\frac{d^{n-1}y}{dx^{n-1}}$, then $\frac{d^ny}{dx^n}=\frac{dp}{dx}$, and (1) becomes

$$f\left(\frac{dp}{dx},\ p,\ x\right)=0,$$

an equation of the first order between p and x. Its solution gives p in terms of x; thus,

$$p = \frac{d^{n-1}y}{dx^{n-1}} = F(x).$$

The value of y can be found from this by successive integration.

Ex. 1. Find the curve whose radius of curvature is constant and equal to R.

The expression of the given condition gives

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = R.$$

Substituting p for $\frac{dy}{dx}$, clearing of fractions, and separating the variables,

$$\frac{dp}{(1+p^2)^{\frac{3}{2}}} = \frac{dx}{R}.$$

Integrating, $$\frac{p}{\sqrt{1+p^2}} = \pm\frac{x-a}{R},$$

in which a is an arbitrary constant of integration.

Solving for p, $$p = \frac{dy}{dx} = \pm\frac{x-a}{\sqrt{R^2-(x-a)^2}},$$

whence, $$y - b = \pm\sqrt{R^2-(x-a)^2},$$

in which b is the arbitrary constant of integration. This result may be written

$$(x-a)^2 + (y-b)^2 = R^2.$$

The integral represents all circles of radius R.

Ex. 2. Solve $\frac{d^2y}{dx^2} - a\left(\frac{dy}{dx}\right)^2 = 0.$

Ex. 3. Solve $\frac{d^3y}{dx^3}\frac{d^2y}{dx^2} = 2.$

112. **Equations of the second order with one variable absent.** (*a*) Equations of the form

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0, \tag{1}$$

on the substitution of p for $\frac{dy}{dx}$, become

$$f\left(\frac{dp}{dx},\ p,\ x\right)=0, \tag{2}$$

an equation of the first order in p and x. Suppose that the solution of (2) is

$$p=\frac{dy}{dx}=F(x,\ c_1).$$

Then the solution of (1) is $y=\int F(x,\ c_1)\,dx+c_2$.

(*b*) Equations of the form $f\left(\frac{d^2y}{dx^2},\ \frac{dy}{dx},\ y\right)=0,$ (3)

on the substitution of p for $\frac{dy}{dx}$, become

$$f\left(p\frac{dp}{dy},\ p,\ y\right)=0,$$

an equation of the first order in p and y. Suppose that its solution is

$$p=\frac{dy}{dx}=F(y,\ c_1).$$

Then the solution of (3) is

$$\int\frac{dy}{F(y,\ c_1)}=x+c_2.$$

Equations of the type in Art. 110 and Ex. 1, Art. 111, are examples of the equations discussed in this article.

Ex. 1. Solve $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0.$

Ex. 2. Solve $\frac{d^4y}{dx^4}+a^2\frac{d^2y}{dx^2}=0.$

Ex. 3. Solve $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1.$

113. Linear equations. General properties. Complementary Function. Particular Integral.

The form of the linear equation of order n is

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_{n-1}\frac{dy}{dx}+P_ny=X, \qquad (1)$$

where $P_1, P_2, \cdots, P_n, X$, are constants or functions of x. The linear equation of the first order was considered in Art. 102.

The complete integral of

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy}{dx}+P_ny=0 \qquad (2)$$

is contained in the complete solution of (1). If $y=f_1(x)$ be an integral of (2), then, as may be seen on substitution, $y=c_1f_1(x)$, c_1 being an arbitrary constant, is also an integral. Similarly, if $y=f_2(x)$, $y=f_3(x)$, $\cdots$, $y=f_n(x)$, be integrals of (2), then $y=c_2f_2(x)$, $\cdots$, $y=c_nf_n(x)$, wherein $c_2, \cdots, c_n$, are arbitrary constants, are integrals of (2). Moreover, substitution will show that

$$y=c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x) \qquad (3)$$

is an integral. If $f_1(x), f_2(x), \cdots, f_n(x)$, are linearly independent, (3) is the complete integral of (2), since it contains n arbitrary constants.

If $\qquad y=F(x)$

be a solution of (1), then $\qquad y=Y+F(x), \qquad (4)$

in which $\qquad Y=c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)$,

is also a solution of (1). For, the substitution of Y for y in the first member of (1) gives zero, and that of $F(x)$ for y, by hypothesis, gives X. Since the solution (4) contains n arbitrary constants, it is the complete solution of (1). The part Y is called *the complementary function*, and the part $F(x)$ is called *the particular integral*. Equations of the form (2) will be considered in the articles that follow.

114. The linear equation with constant coefficients and second member zero. On the substitution of e^{mx} for y, the first member of

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = 0 \tag{1}$$

becomes $$(m^n + P_1m^{n-1} + \cdots + P_{n-1}m + P_n)\, e^{mx}.$$

This expression is equal to zero if

$$m^n + P_1m^{n-1} + \cdots + P_{n-1}m + P_n = 0. \tag{2}$$

The latter may be called the *auxiliary equation.* Therefore, if m_1 be a root of (2), $y = e^{m_1x}$ is an integral of (1); and if the n roots of (2) be $m_1, m_2, \cdots, m_n$, the complete solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}. \tag{3}$$

Ex. 1. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 35\,y = 0.$

The auxiliary equation is $m^2 - 2\,m - 35 = 0$;

and its roots are $m = -5,\ m = 7.$

Hence, the complete solution is

$$y = c_1e^{-5x} + c_2e^{7x}.$$

If the auxiliary equation has a pair of *imaginary roots,* say $m_1 = a + i\beta$, $m_2 = a - i\beta$ (i denoting $\sqrt{-1}$), the corresponding part of the integral can be put in a real form. For,

$$\begin{aligned} c_1e^{(a+i\beta)x} + c_2e^{(a-i\beta)x} &= e^{ax}\,(c_1e^{i\beta x} + c_2e^{-i\beta x}) \\ &= e^{ax}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\} \\ &= e^{ax}(A\cos\beta x + B\sin\beta x). \end{aligned}$$

Ex. 2. Solve $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25\,y = 0.$

The auxiliary equation is

$$m^2 + 8\,m + 25 = 0,$$

and its roots are $m = -4 + 3\,i,\ m = -4 - 3\,i.$

Hence, the integral is $y = e^{-4x}(c_1\cos 3\,x + c_2\sin 3\,x).$

Ex. 3. Solve $2\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} - 12x = 0.$

Ex. 4. Solve $\dfrac{d^3y}{dx^3} + 8y = 0.$

Ex. 5. Solve $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0.$

Ex. 6. Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 0.$

Ex. 7. Solve $\dfrac{d^4y}{dx^4} - a^4y = 0.$

115. Case of the auxiliary equation having equal roots. If two roots of (2) Art. 114 are *equal*, say m_1 and m_2, solution (3) becomes

$$y = (c_1 + c_2)\, e^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

Since $c_1 + c_2$ is equivalent to a single constant, this solution has $n - 1$ arbitrary constants, and hence, it is not the general solution. In order to obtain the complete solution in this case, make the substitution

$$m_2 = m_1 + h.$$

The terms of the solution that correspond to m_1, m_2, will then be

$$y = c_1e^{m_1x} + c_2e^{(m_1+h)x},$$

that is, $$y = e^{m_1x}\,(c_1 + c_2e^{hx}).$$

On expanding e^{hx} in the exponential series, this becomes

$$y = e^{m_1x}\left[c_1 + c_2\left(1 + hx + \frac{h^2x^2}{1\cdot 2} + \frac{h^3x^3}{1\cdot 2\cdot 3} + \cdots\right)\right]$$

$$= e^{m_1x}\left[c_1 + c_2 + c_2\,hx\left(1 + \frac{hx}{1\cdot 2} + \frac{h^2x^2}{1\cdot 2\cdot 3} + \cdots\right)\right]$$

$$= e^{m_1x}(A + Bx + \tfrac{1}{2}\,c_2h^2x^2 + \text{terms in ascending powers of } h),$$

in which $A = c_1 + c_2$, and $B = c_2h$.

Now let h approach zero, and solution (3), Art. 114, takes the form

$$y = e^{m_1x}\,(A + Bx) + c_3e^{m_3x} + \cdots + c_ne^{m_nx};$$

for the arbitrary constants c_1, c_2 can be chosen so that A, B will be finite, and that c_2h^2, etc., will approach zero. If the auxiliary equation have three roots equal to m_1, it can be shown that the corresponding solution is

$$y = e^{m_1x}(c_1 + c_2x + c_3x^2),$$

and, if it have r equal roots, that the corresponding solution is

$$y = e^{m_1x}(c_1 + c_2x + \cdots + c_{r-1}x^{r-1}).$$

If a pair of imaginary roots, $\alpha + i\beta$, $\alpha - i\beta$, occurs twice, the corresponding solution is

$$y = (c_1 + c_2x)e^{(\alpha+i\beta)x} + (c_3 + c_4x)e^{(\alpha-i\beta)x},$$

which reduces to

$$y = e^{\alpha x}\{(A + A_1x)\cos\beta x + (B + B_1x)\sin\beta x\}.$$

Ex. 1. Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$

The auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0,$$

of which the roots are $+1$ repeated three times. Hence, the solution is

$$y = e^x(c_1 + c_2x + c_3x^2).$$

Ex. 2. Solve $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

Ex. 3. Solve $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} - 9\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 4y = 0.$

116. The homogeneous linear equation with the second member zero. This equation has the form

$$x^n\frac{d^ny}{dx^n} + p_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + p_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}x\frac{dy}{dx} + p_ny = 0, \qquad (1)$$

wherein $p_1, p_2, \cdots, p_n$, are constants.

(*a*) *First method of solution.* If the substitution

$$z = \log x, \quad \text{that is,} \quad x = e^z,$$

be made, equation (1) will be transformed into

$$\frac{d^ny}{dz^n} + q_1 \frac{d^{n-1}y}{dz^{n-1}} + q_2 \frac{d^{n-2}y}{dz^{n-2}} + \cdots + q_{n-1} \frac{dy}{dz} + q_n y = 0, \tag{2}$$

wherein $q_1, q_2, \cdots, q_n$, are constants. This can be easily verified.

Ex. 1. Solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$.

If $z = \log x$, then $\frac{dz}{dx} = \frac{1}{x}$, and

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dz}\right).$$

Substitution of these values in the given equation changes it into

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + 1 = 0,$$

the solution of which is $y = e^z(c_1 + c_2 z)$,

whence, $y = x(c_1 + c_2 \log x)$.

(*b*) *Second method of solution.* The substitution of x^m for y in the first member of (1) gives

$$\{m(m-1)\cdots(m-n+1) + p_1 m(m-1)\cdots(m-n+2) + \cdots + p_{n-1}m + p_n\}x^m.$$

This is zero, and accordingly $y = x^m$ is a solution of (1), if

$$m(m-1)\cdots(m-n+1) + p_1 m(m-1)\cdots(m-n+2) + \cdots + p_{n-1}m + p_n = 0. \tag{3}$$

Hence, if $m_1, m_2, \cdots, m_n$, are the roots of (3), the complete solution of (1) is

$$y = c_1 x^{m_1} + c_2 x^{m_2} + \cdots + c_n x^{m_n}.$$

To a solution $y = e^{mz}(c_1 + c_2 z + \cdots + c_{r-1}x^{r-1})$ of (2), there corresponds a solution $y = x^m(c_1 + c_2 \log x + \cdots + c_{r-1}(\log x)^{r-1})$ of (1), since $z = \log x$. It can be easily shown that the auxiliary equation of (2) is identical with (3). Hence, if m_1 is repeated r times as a root of (3), the corresponding solution of (1) is

$$y = x^{m_1}\{c_1 + c_2 \log x + \cdots + c_{r-1}(\log x)^{r-1}\}.$$

Ex. 2. Solve $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0.$

Substitution of x^m for y gives $(m^3 + 1)x^m = 0,$

of which the roots are $-1, \ \dfrac{1+\sqrt{3}\,i}{2}, \ \dfrac{1-\sqrt{3}\,i}{2}.$

Hence the solution is

$$y = \frac{c_1}{x} + x^{\frac{1}{2}}\left\{c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right\}.$$

Ex. 3. $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0.$

Ex. 4. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 0.$

Ex. 5. $x^3 \dfrac{d^3y}{dx^3} - 3x^2 \dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} - 8y = 0.$

EXAMPLES

1. If $y = Ae^{kx} + Be^{-kx}$, prove that $\dfrac{d^2y}{dx^2} - k^2y = 0.$

2. Derive the differential equation of all circles which pass through the origin and whose centers are on the x-axis.

3. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0.$

4. $x\,dx + y\,dy = a^2 \dfrac{x\,dy - y\,dx}{x^2 + y^2}.$

5. $\dfrac{dx}{x^2 - 2xy} = \dfrac{dy}{y^2 - 2xy}.$

6. $(2ax + by + g)\,dx + (2cy + bx + e)\,dy = 0.$

7. $(1 + xy)\,y\,dx + (1 - xy)x\,dy = 0.$

8. $y(2xy + e^x)\,dx - e^x\,dy = 0.$

9. $x\dfrac{dy}{dx} - ay = x + 1.$

10. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$

11. $x(1-x^2)\dfrac{dy}{dx}+(2\,x^2-1)\,y=ax^3.$

12. $\dfrac{dy}{dx}+y\cos x=y^n\sin 2\,x.$

13. $\dfrac{dy}{dx}=x^3y^3-xy.$

14. $p^3(x+2\,y)+3\,p^2(x+y)+(y+2\,x)\,p=0.$

15. $xp^2-2\,yp+ax=0.$

16. $p^2y+2\,px=y.$

17. $e^{3x}(p-1)+p^3e^{2y}=0.$

18. $y^3\dfrac{d^2y}{dx^2}=a.$

19. $\dfrac{d^4y}{dx^4}-a^2\dfrac{d^2y}{dx^2}=0.$

20. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^3=0.$

21. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^x.$

22. $\dfrac{d^3y}{dx^3}+y=0.$

23. $\dfrac{d^4y}{dx^4}+2\,n^2\dfrac{d^2y}{dx^2}+n^4y=0.$

24. $\dfrac{d^4y}{dx^4}-4\dfrac{d^3y}{dx^3}+8\dfrac{d^2y}{dx^2}-8\dfrac{dx}{dy}+4\,y=0.$

25. $\dfrac{d^4y}{dx^4}-2\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}=0.$

26. $x^3\dfrac{d^3y}{dx^3}+2\,x^2\dfrac{d^2y}{dx^2}+2\,y=0.$

27. $x^4\dfrac{d^4y}{dx^4}+6\,x^3\dfrac{d^3y}{dx^3}+9\,x^2\dfrac{d^2y}{dx^2}+3\,x\dfrac{dy}{dx}+y=0.$

28. $\dfrac{P}{2}\left(\dfrac{l}{2}-x\right)-EI\dfrac{d^2y}{dx^2}=0.$

29. $EI\dfrac{d^4y}{dx^4}=-\gamma bx.$

APPENDIX

NOTE A

[This note is supplementary to Art. 33]

A method of decomposing a rational fraction into its partial fractions.

Suppose that $\frac{f(x)}{F(x)(x-a)^p}$ is a proper rational fraction. The substitution of a for x in this fraction, $(x-a)^p$ being left unchanged, and the subtraction of the fraction thus formed gives

$$\frac{f(x)}{F(x)(x-a)^p}-\frac{f(a)}{F(a)(x-a)^p}=\frac{f(x)F(a)-f(a)F(x)}{F(a)F(x)(x-a)^p}. \qquad (1)$$

The numerator of the fraction in the second member of (1) vanishes for $x=a$, and hence it is divisible by $x-a$. Let the quotient be denoted by $\phi(x)$. Then

$$\frac{f(x)}{F(x)(x-a)^p}=\frac{f(a)}{F(a)(x-a)^p}+\frac{1}{F(a)}\cdot\frac{\phi(x)}{F(x)(x-a)^{p-1}}. \qquad (2)$$

Of the two fractions in the second member of (2) the first is one of the partial fractions required, and the second has a denominator of lower degree than the original fraction possesses. The second fraction can be similarly decomposed, and by the repetition of the operation all the partial fractions will be found.

When the factors of the denominator are all different and of the first degree, the decomposition of a fraction can be effected very quickly. For example, on taking the fraction

$\dfrac{f(x)}{(x-a)(x-b)(x-c)}$ and substituting a for x except in $x-a$, and subtracting the fraction thus formed, there is obtained

$$\frac{f(x)}{(x-a)(x-b)(x-c)}-\frac{f(a)}{(x-a)(a-b)(a-c)}=\frac{F(x)}{(x-b)(x-c)},$$

in which $F(x)$ is a constant or of the first degree in x.

The partial fraction whose denominator is $x-a$, which is formed by this rule, is accordingly $\dfrac{f(a)}{(x-a)(a-b)(a-c)}$. The partial fractions whose denominators are $(x-b)$, $(x-c)$, can be written by symmetry. This is easily verified. For, on assuming

$$\frac{f(x)}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c},$$

and clearing of fractions,

$$f(x)=A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b).$$

The substitution of a for x gives

$$f(a)=A(a-b)(a-c),$$

whence,
$$A=\frac{f(a)}{(a-b)(a-c)}.$$

It will be found, on putting b for x that

$$B=\frac{f(b)}{(b-a)(b-c)},$$

and on putting c for x, that

$$C=\frac{f(c)}{(c-a)(c-b)}.$$

Therefore,

$$\frac{f(x)}{(x-a)(x-b)(x-c)}=\frac{f(a)}{(x-a)(a-b)(a-c)}$$
$$+\frac{f(b)}{(b-a)(x-b)(b-c)}+\frac{f(c)}{(c-a)(c-b)(x-c)}.$$

Ex. 1.

$$\frac{2x^2-1}{(x-2)(x+3)(x-5)} = \frac{2\cdot 2^2-1}{(x-2)(2+3)(2-5)} + \frac{2(-3)^2-1}{(-3-2)(3+x)(-3-5)}$$

$$+\frac{2\cdot 5^2-1}{(5-2)(5+3)(x-5)}$$

$$= -\frac{7}{15(x-2)} + \frac{17}{40(x+3)} + \frac{49}{24(x-5)}.$$

Ex. 2. $\frac{3x+2}{(x-2)^2(x-3)} - \frac{3\cdot 2+2}{(x-2)^2(2-3)} = F.$

On determining F by subtraction,

$$\frac{3x+2}{(x-2)^2(x-3)} + \frac{8}{(x-2)^2} = \frac{11}{(x-2)(x-3)}$$

$$= \frac{11}{(3-2)(x-3)} + \frac{11}{(x-2)(2-3)}.$$

Hence, $$\frac{3x+2}{(x-2)^2(x-3)} = -\frac{8}{(x-2)^2} + \frac{11}{x-3} - \frac{11}{x-2}.$$

NOTE B

[This note is supplementary to Art. 45]

To find reduction formulæ for $\int x^m(a+bx^n)^p\,dx$ by integration by parts. In what follows $\int x^m(a+bx^n)^p\,dx$ will be denoted by I.

(*a*) On putting $dv = x^{n-1}(a+bx^n)^p\,dx,\quad u = x^{m-n+1},$

it follows that $v = \frac{(a+bx^n)^{p+1}}{nb(p+1)},\quad du = (m-n+1)\,x^{m-n}\,dx.$

Hence, $$I = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}\,dx.$$

But $x^{m-n}(a+bx^n)^{p+1} = x^{m-n}(a+bx^n)(a+bx^n)^p$

$$= ax^{m-n}(a+bx^n)^p + bx^m(a+bx^n)^p.$$

Therefore

$$I=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb\,(p+1)}-\frac{m-n+1}{nb\,(p+1)}\left[\int ax^{m-n}(a+bx^n)^p\,dx+bI\right].$$

On solving for I,

$$I=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b\,(m+np+1)}-\frac{a\,(m-n+1)}{b\,(m+np+1)}\int x^{m-n}(a+bx^n)^p\,dx.$$

(*b*) On transposition in the result just obtained and division by $\dfrac{a\,(m-n+1)}{b\,(m+n+p+1)}$,

$$\int x^{m-n}(a+bx^n)^p\,dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{a\,(m-n+1)}$$

$$-\frac{b\,(m+np+1)}{a\,(m-n+1)}\int x^m(a+bx^n)^p\,dx.$$

From this, on changing m into $m+n$,

$$\int x^m(a+bx^n)^p\,dx=\frac{x^{m+1}(a+bx^n)^{p+1}}{a\,(m+1)}$$

$$-\frac{b\,(m+n+np+1)}{a\,(m+1)}\int x^{m+n}(a+bx^n)^p\,dx.$$

(*c*) On putting $dv=x^m\,dx, \qquad u=(a+bx^n)^p,$

it follows that $\qquad v=\dfrac{x^{m+1}}{m+1}, \quad du=pnbx^{n-1}(a+bx^n)^{p-1}\,dx$

and hence,

$$I=\frac{x^{m+1}\,(a+bx^n)^p}{m+1}-\frac{pnb}{m+1}\int x^{m+n}\,(a+bx^n)^{p-1}\,dx.$$

But $x^{m+n}=x^mx^n=\dfrac{x^m\,(a+bx^n)}{b}-\dfrac{ax^m}{b}.$

On substituting this value in the last integral,

$$I=\frac{x^{m+1}(a+bx^n)^p}{m+1}-\frac{pnb}{m+1}\left[\frac{I}{b}-\frac{a}{b}\int x^m\,(a+bx^n)^{p-1}\,dx\right].$$

Whence, on solving for I,

$$I = \frac{x^{m+1}(a+bx^n)^p}{m+np+1} + \frac{anp}{m+np+1}\int x^m (a+bx^n)^{p-1}\,dx.$$

On transposing in the last result and dividing by $\dfrac{anp}{m+np+1}$,

$$\int x^m(a+bx^n)^{p-1}\,dx = -\frac{x^{m+1}(a+bx^n)^p}{anp} + \frac{m+np+1}{anp}\int x^m(a+bx^n)^p\,bx.$$

From this, on changing p into $p+1$,

$$\int x^m(a+bx^n)^p\,dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)}$$
$$+\frac{m+n+np+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}\,dx.$$

NOTE C

[This note is supplementary to Art. 51]

To find reduction formulæ for $\int \sin^m x \cos^n x\,dx$ by integration by parts. On denoting this integral by I,

$$I = \int \sin^m x \cos^n x\,dx = -\int \sin^{m-1} x \cos^n x\,d(\cos x).$$

On putting $dv = \cos^n x\,d(\cos x),\; u = \sin^{m-1} x,$

it follows that $v = \dfrac{\cos^{n+1} x}{n+1},\; du = (m-1)\sin^{m-2} x \cos x\,dx.$

Hence, $$I = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x\,dx.$$

But $$\sin^{m-2} x \cos^{n+2} x = \sin^{m-2} x \cos^n x\,(1-\sin^2 x)$$
$$= \sin^{m-2} x \cos^n x - \sin^m x \cos^n x.$$

Hence, $$I = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\left[\int \sin^{m-2} x \cos^n x\,dx - I\right].$$

From this, on solving for I,

$$I = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx.$$

From this result, on transposition and division by $\frac{m-1}{n+1}$,

$$\int \sin^{m-2} x \cos^n x \, dx = \frac{\sin^{m-1} x \cos^{n+1} x}{m-1} + \frac{m+n}{m-1} \int \sin^m x \cos^n x \, dx;$$

whence, on changing m into $m+2$,

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x \, dx.$$

Formula C, Art. 51, can be obtained by writing

$$I = \int \cos^{n-1} x \sin^m x \, d(\sin x),$$

putting $\quad dv = \sin^m x \, d(\sin x),\ u = \cos^{n-1} x,$

and then integrating by parts and reducing.

Formula D, Art. 51, can be derived from C by transposition and the change of n into $n+2$.

NOTE D

[This note is supplementary to Art. 67]

It is explained in the differential calculus that if the difference between two quantities be infinitesimal compared with either of them, then the limit of their ratio is unity, and either of them can be replaced by the other in any expression involving the quantities. A deduction that can be made by means of this principle is of great importance in the practical applications of the integral calculus.

If $$\alpha_1 + \alpha_2 + \cdots + \alpha_n$$

represent the sum of a number of infinitesimal quantities which approaches a finite limit as n is increased indefinitely, and if $$\beta_1,\ \beta_2,\ \cdots,\ \beta_n$$

be another system of infinitesimal quantities, such that

$$\frac{\beta_1}{\alpha_1} = 1 + e_1,\quad \frac{\beta_2}{\alpha_2} = 1 + e_2,\quad \cdots,\quad \frac{\beta_n}{\alpha_n} = 1 + e_n, \tag{1}$$

where $$e_1,\ e_2,\ \cdots,\ e_n$$

are infinitesimal quantities, then the limit of the sum of $\beta_1, \beta_2, \cdots, \beta_n$ is equal to the limit of the sum of $\alpha_1, \alpha_2, \cdots, \alpha_n$.

It follows from equations (1) that

$$\beta_1 + \beta_2 + \cdots + \beta_n = (\alpha_1 + \alpha_2 + \cdots + \alpha_n) + (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n). \tag{2}$$

Let η be one of the infinitesimal quantities $e_1, e_2, \cdots, e_n$, which is not less than any one of the others. Then

$$(\beta_1 + \beta_2 + \cdots + \beta_n) - (\alpha_1 + \alpha_2 + \cdots + \alpha_n) < (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\,\eta.$$

But by hypothesis $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ has a finite limit, and hence the second member of this inequality is infinitesimal. Therefore the limit of $\beta_1 + \beta_2 + \cdots \beta_n$ is the same as the limit of $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.*

NOTE E

[This note is supplementary to Arts. 84–87]

Further rules for the approximate determination of areas. A few more rules for approximately determining the area of *LAKT* (Fig. 53) may be stated. As before, h denotes the interval between successive equidistant ordinates, and merely the coefficients of the successive ordinates are given in the formulæ. In the trapezoidal rule, strips were taken in which two ordinates were drawn; in other words, the ordinates were taken by twos. In the parabolic rule, strips were taken in which three ordinates were drawn; that is, the ordinates were taken by threes.

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* See B. Williamson, *Treatise on the Differential Calculus*, Arts. 38–40.

Rule A. If four ordinates are taken at a time, then for each strip, approximately,

$$\textbf{area} = \tfrac{3}{8}\boldsymbol{h}\,(\mathbf{1} + \mathbf{3} + \mathbf{3} + \mathbf{1}).$$

This is commonly called Simpson's three-eighths rule.

Rule B. If five ordinates are taken at a time, then approximately for a strip involving them,

$$\text{area} = \tfrac{2}{45}h\,(7 + 32 + 12 + 32 + 7).$$

Rule C. If six ordinates are taken at a time,

$$\text{area} = \tfrac{125}{288}h\,(\tfrac{19}{25} + 3 + 2 + 2 + 3 + \tfrac{19}{25}).$$

Rule D. If seven ordinates are taken at a time, then, after a very slight modification of the formula that first presents itself,

$$\textbf{area} = \tfrac{3}{10}\boldsymbol{h}\,(\mathbf{1} + \mathbf{5} + \mathbf{1} + \mathbf{6} + \mathbf{1} + \mathbf{5} + \mathbf{1}).$$

This is known as Weddle's* rule. It may be expressed in words: The proposed area being divided into six portions by seven equidistant ordinates, to five times the sum of the even ordinates add the middle ordinate and all the odd ordinates, multiply the sum by three tenths of the common interval, and the product will be the required area approximately. Rules A and D are frequently employed.

The trapezoidal and parabolic rules A, B, C, above, are special cases of one general rule† which is deduced on the supposition that the area concerned is divided into n portions bounded by $n+1$ equidistant ordinates whose lengths and common distance apart are known. The given curve, say $y = \phi(x)$, is replaced by a curve which passes through the extremities of the $n+1$ given ordinates, and whose equation is a rational integral function of x of the nth degree. The area of the latter curve can be easily found by integration. For example, in the case of each successive pair of ordinates in Art. 85, the given arc was replaced by a straight line, and in the case of each successive group of three ordinates in Art. 86, the given arc was replaced by the arc of a parabola. On assuming that the equation of the second curve is

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* It was first given by Mr. Thomas Weddle in the Cambridge and Dublin Mathematical Journal, Vol. IX. (1854), pp. 79, 80.

† This rule was first given by Newton and Cotes, and published by the latter in 1722 in a tract, *De Methodo Differentiali.*

$$y = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n, \tag{1}$$

the coefficients A_0, A_1, A_2, $\cdots$, A_n, can be determined. For, the substitution in (1) of the coördinates of the $n+1$ given points, namely the extremities of the given equidistant ordinates, will give $n+1$ equations, by means of which the values of the $n+1$ coefficients A_0, A_1, $\cdots$, A_n, can be found.* If n is sufficiently great, the difference between the area of the second curve and that of the original curve will generally be very small. The general formula for the case of $n+1$ equidistant ordinates can also be deduced by the method of finite differences.† For a discussion on various methods of finding an approximate value of a definite integral by numerical calculation, reference may be made to J. Bertrand, *Calcul Intégral*, Chapter XII., pp. 331–352.

NOTE F

[This note is supplementary to Art. 88]

The Fundamental Theory of the Planimeter ‡

In Fig. 58, $ALBC$ is a plane figure whose area is required, and QX is a given straight line taken as the axis of X. Let MN represent a plate of which two given points always move, Q along QX, and P on the contour of the given area. Then QP is a straight line fixed with reference to the instrument. Let b be the length of QP. Let W be the recording wheel with axis parallel to QP. Its actual location is arbitrary.

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* Also see Lamb, *Infinitesimal Calculus*, Art. 112.

† See Boole, *Calculus of Finite Differences*, Chapter III, Arts. 10–14.

‡ This note is by Professor W. F. Durand, who has kindly permitted its insertion here.

The movement of P from P to C, a point very near, may be decomposed into: (1) A movement dx parallel to QX, (2) a movement dy at right angles to QX. It will first be shown that the record of the wheel W due to the dy component will, for the closed area, be zero.

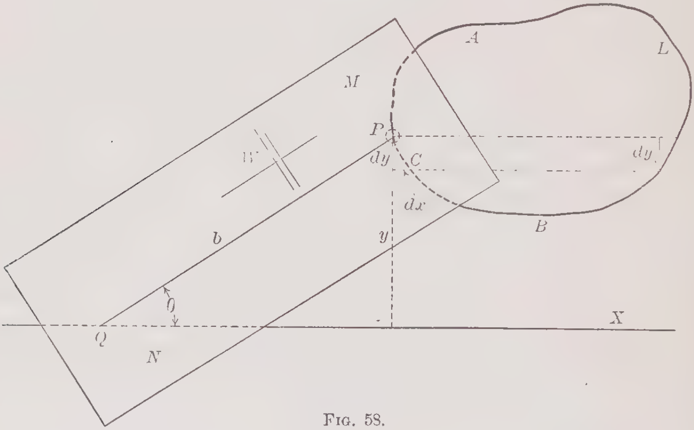


FIG. 58.

It may be noted that the amount of the dy record depends on the dy and on the configuration of the instrument under which it is traversed. Now it is evident that for every dy traversed in the *up* direction, there will be an equal dy traversed in the *down* direction, and under the same configuration. In the diagram the pair thus traversed is dy and dy_1. The net record for such a pair is zero, and for every other pair, zero, and therefore, for the entire contour, zero.

It follows that the entire record will be merely that due to the dx components. This is found as follows. The component of dx in the direction of the plane of the wheel is $dx \cdot \sin\theta$. But $\sin\theta = \frac{y}{b}$. Denote that part of the record due to dx by dR. Then,

$$dR = dx \cdot \sin\theta = \frac{y\,dx}{b}.$$

Hence, $$R = \frac{1}{b}\int y\,dx = \frac{A}{b},$$

and therefore $$A = bR.$$

It only remains therefore to graduate W conformably to the length of b, or, *vice versa*, to graduate W and give to b an appropriate length. The latter is the usual method. By giving to b various lengths, the area may be read off in corresponding units.

Thus far it has been assumed that Q follows the straight line QX. It will next be shown that the record is independent of the path of Q so long as it is back and forth on the same line.

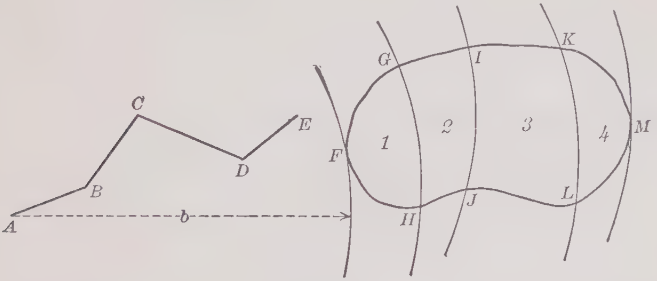


FIG. 59.

To this end let FM (Fig. 59) be any area, and $ABCDE$ a broken line. Let A and E be the points from which arcs with a radius b will be tangent to the contour at F and M. With the same radius and B, C, D as centers, draw arcs as shown in the figure. Suppose now that P is carried around these partial areas successively, and always in the same cyclical direction. For (1) the point Q (Fig. 58) will traverse AB, for (2), BC, etc. In each single case the record will represent the corresponding area. Therefore the total area will be represented by the total record. And it is readily seen that GH, IJ, KL, are each traversed twice in opposite directions. Hence the record due to them is zero, and the actual record is due only to the external contour. Hence, if P were carried directly around the external contour, it should have the same record, and hence the area. This is true for any broken line, and hence for a curve. In the common polar planimeter the curve is the arc of a circle. Thus the point C in Fig. 55, Art. 88, which corresponds to the point Q moving along QX in Fig. 58, or along $ABCDE$ in Fig. 59, moves along the circumference of the circle of center T and radius a.

NOTE G

ON INTEGRAL CURVES

1. Applications to mechanics. — (This article is supplementary to Art. 93.)

(a) The statical moment about $OY = M_y =$ (area) $OH = ay_1\overline{OH}$.

Hence,

$$M_y = ay_1(x_1 - HX) = ay_1x_1 - aby_2 = a(y_1x_1 - by_2). \tag{1}$$

Also

$$I_{GH} = I_{PX} - \text{(area)}\,(HX)^2 = 2\,abcy_3 - \frac{ab^2y_2^2}{y_1} = ab\left(2\,cy_3 - \frac{by_2^2}{y_1}\right). \tag{2}$$

(b) The value of HX in Art. 93 (b), may be found by a simple construction, though from its nature the accuracy may not be all that is desirable.

Let BH be drawn tangent to OB at B. Then

$$\tan BHX = \frac{dy_2}{dx} = \frac{y_2}{HX}.$$

But $$y_2 = \frac{1}{b}\int y_1\,dx, \text{ and hence } \frac{dy_2}{dx} = \frac{y_1}{b}.$$

Hence, $$\frac{y_2}{HX} = \frac{y_1}{b}, \text{ and } HX = \frac{by_2}{y_1}.$$

Hence, from Art. 93 (b), the point H thus determined will be the abscissa of the center of gravity as desired.

(c) The moment of any area $ORFH$ about any vertical XA is proportional to the corresponding ordinate XD of the tangent to the second integral curve at the point E on the limiting ordinate HF.

It has been shown in Art. 91 that $\frac{dy_2}{dx} = \frac{1}{b} y_1$. From this,

$$\tan DKX = \frac{EH}{HK} = \frac{dy_2}{dx} = \frac{y_1}{b} = \frac{\text{area}}{ab}.$$

Hence the equation to the tangent KD is

$$y - HE = \frac{A}{ab}(x - OH);$$

whence,
$$aby = ab \cdot HE + A(x - OH). \tag{3}$$

But from Art. 93, $ab \cdot HE$ is the moment of the area about HF, and $A(x - OH)$ is the correction necessary to transfer this moment to an axis distant $(x - OH)$ from HF, and therefore distant x from the origin. The second member of (3) is thus seen to be equal to the moment of the area about a vertical line at any distance x from the origin. Hence, such moment is measured by aby, or ab times that ordinate of the tangent line which is determined by the abscissa x. Hence such ordinate at any point bears the same relation to the moment of $ORFH$ about the vertical line containing the ordinate, that HE does to the moment about HF.

(*d*) It follows that where KD crosses OX, the moment about the corresponding ordinate will be zero, and hence an ordinate through K will contain the center of gravity of the area $ORFH$. Hence the construction given in (*b*) above is a special case of (*c*).

(*e*) If we apply the same proposition to the moments of the two areas $ORFH$ and $ORPX$ about an ordinate through L, the point of intersection of the two tangents at E and B, we shall have for each moment the expressions $abNL$, and the moment of the difference of the two areas or of $HFPX$ about NS will be zero, and therefore NS will contain the center of gravity of such area.

Hence the tangents to the second integral curve at any two ordinates intersect on the ordinate which contains the center of gravity of the area of the fundamental curve lying between the two ordinates chosen.

2. Applications in engineering and in electricity. The limits of the present article will not allow detailed reference to the various ways in which these curves may be made of use in studying engineering problems. A few brief references may, however, be made to some of the more common applications.

It is readily seen by comparison with text-books on mechanics that if for the fundamental we take the curve of net external force on a beam or girder, then the first integral of such fundamental will give the entire history of the shear from one end to the other. Also that the second integral will give similarly the entire history of the bending moment from one end to the other. This serves to illustrate one important advantage of representation by means of these curves, and that is, that they serve to give not only the value at some one or more desired points, but at all points as well.

In this way they furnish a continuous history of the variation of the function in question, and thus give a far more vivid picture of its characteristics than can be obtained in any other way. In the case of beams or girders it may be well to note that external forces should not be assumed as concentrated at a point, but should rather be considered as distributed over a length equal to that occupied by the object to the existence of which they are due. Thus the supporting forces at the ends of a bridge span must not be considered as located at a point, as is common in the analytical treatment, but rather as distributed over a length equal to that occupied by the supporting pier. Their graphical representation will therefore be a rectangle, or rather it may be so taken for all practical purposes.

As another application, consider the action of a varying effort or force acting through moving parts having inertia, and upon a dissimilarly varying resistance, their mean values being of course the same. This is the case with the ordinary steam engine or other prime mover operating against a variable resistance.

Suppose that we have plotted on a distance abscissa, the curves of effort and of resistance. The integral of the first will give the history of the work as done by the agent or effort, while that of the second will give the history of the work as done upon the resistance. Steady conditions being assumed, their mean values will be the same. Their history, however, will be quite different. The difference between the ordinates at any point will give the work stored as energy in the moving parts during their acceleration when the effort is greater than the resistance, restored during their retardation when the effort is less than the resistance. We might reach the same results by taking as our fundamental the difference between the curves of effort and resistance. The integral of this will give the history of the ebb and flow of energy from and into the moving parts of the mechanism. Again, by replotting this latter curve on a time abscissa it becomes representative of the time history of the acceleration of the moving parts. If then the reduced inertia of these parts is known, the acceleration at any instant is known, and the curve may be considered as one of acceleration. Its integral will, therefore, give velocity, such velocity being the increase or decrease above the mean value. Such a curve would, therefore, show the continuous history of variation in the velocity due to the causes mentioned.

In electrical science there are many interesting applications of these methods. Of these only one or two of the simpler will be here given.

Suppose that we have on a time abscissa a curve showing the history of the electromotive force in any circuit. Then since this is the time rate of variation of the total magnetism in the circuit, it is evident that, reciprocally, the latter must be the integral of the former. Hence the first integral curve will give the history of the total magnetic flux in the circuit.

Again, if we have on a time abscissa a curve showing the history of a current, then the history of the growth of the quantity of electricity will be given by the first integral of such curve.

Instances might be widely multiplied, but enough has been given to show that where desired results may be found by one or more integrations effected on a function whose history is known, the complete representation of the problem naturally leads to the production of these curves; and for their practical determination and for their application to many special problems, the fundamental relations and properties as developed above and in Chapter XII., will be found of considerable value.

3. The theory of the integraph. We will next show briefly the fundamental theory of the integraph, an instrument for practically drawing the first integral from its fundamental. Various forms of instrument have been devised, but in nearly all, the kinematic conditions to be fulfilled are the same. These are as

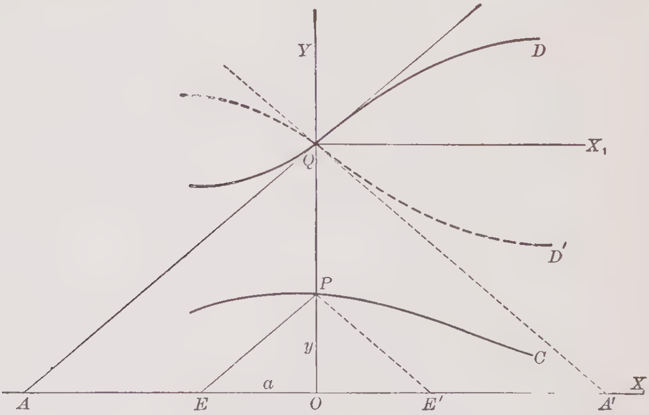


FIG. 60.

follows: Let PC (Fig. 60) be the fundamental relative to axes OX, OY; and QD the integral relative to axes QX_1, QY. For convenience the two Y-axes are taken in the same line, though this is not necessary. P and Q are therefore corresponding points. At Q draw a line QA tangent to QD. From P draw PE parallel to QA. Then:

$$\frac{QO}{OA}=\frac{dy_1}{dx} \text{ and } \frac{PO}{OE}=\frac{y}{a}.$$

Hence,

$$\frac{dy_1}{dx}=\frac{y}{a} \text{ or } y_1=\int\frac{y}{a}dx.$$

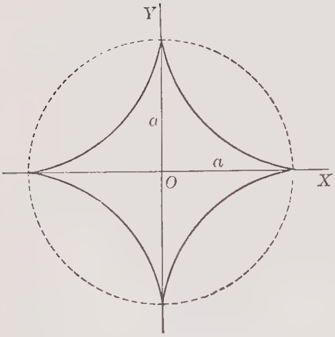
If now a is constant, we shall have

$$y_1=\frac{1}{a}\int y\,dx=\frac{\text{area}}{a} \text{ or area} = ay_1.$$

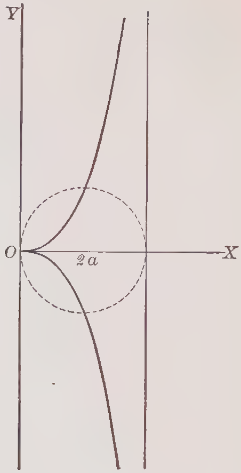
These conditions are seen to correspond to (2), Art. 91. The instrument must therefore include three points, E, P, and Q, related as above specified. While the instrument travels along the direction of X, P is made to trace the given curve, and E remains at a constant distance a from the foot of the ordinate through P. This determines a direction EP, and Q constrained by the structure of the instrument to move always parallel to EP, will trace the integral curve QD.

It is not necessary that the points E and A should lie to the left of O as in Fig. 60. They may be taken as at E', A', and in such case if the fundamental lies above X, the integral will lie below its X as shown by QD', and *vice versa*. The actual values, however, will remain unchanged, and the inversion is readily allowed for in the interpretation of the results.

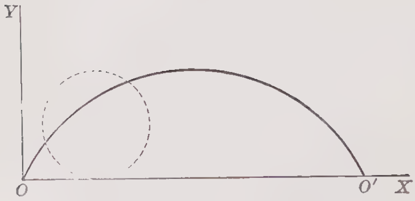
Following are the figures of some of the curves referred to in the preceding pages:



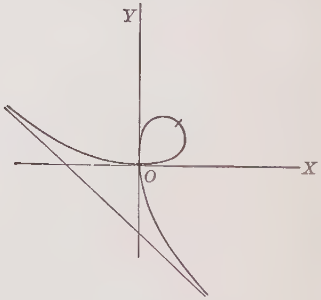
The hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.



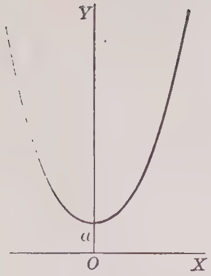
The cissoid, $y^2 = \dfrac{x^3}{2a - x}$.



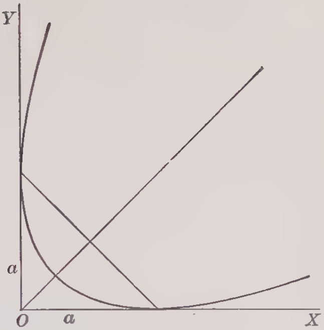
The cycloid, $x = a(\theta - \sin\theta)$;
$y = a(1 - \cos\theta)$.



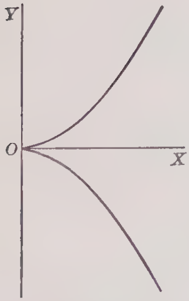
Folium of Descartes,
$x^3 + y^3 = 3axy$.



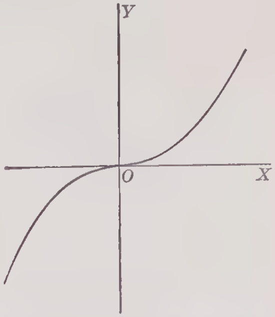
The catenary, $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.



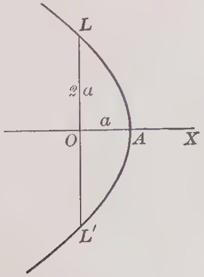
The parabola, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.



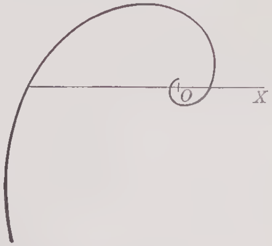
The semicubical parabola, $ay^2 = x^3$.



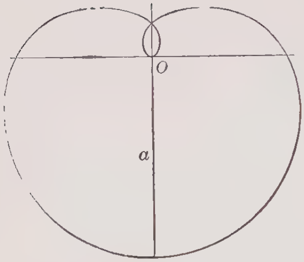
The cubical parabola, $a^2y = x^3$.



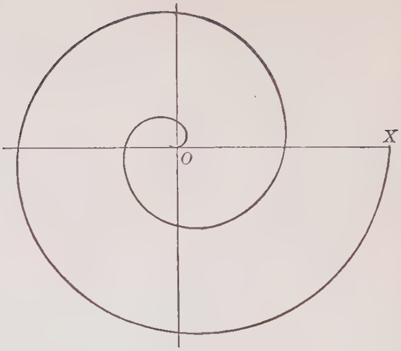
The parabola, $r = a \sec^2 \frac{\theta}{2}$.



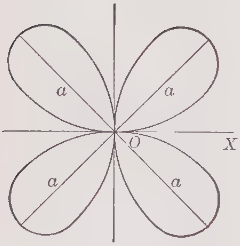
The logarithmic spiral, $r = e^{a\theta}$.



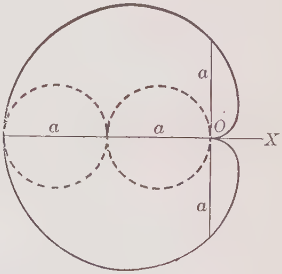
The curve, $r = a \sin^3 \frac{\theta}{3}$.



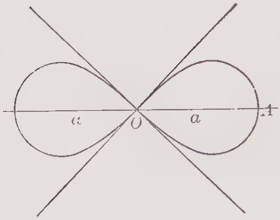
Spiral of Archimedes, $r = a\theta$.



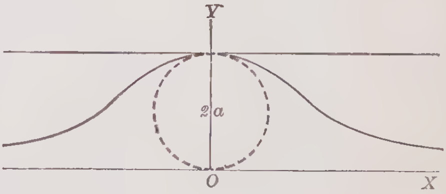
The curve, $r = a \sin 2\theta$.



The cardioid, $r = a(1 - \cos\theta)$.



The lemniscate, $r^2 = a^2 \cos 2\theta$.



The witch, $y = \dfrac{8a^3}{x^2 + 4a^2}$.

A SHORT TABLE OF INTEGRALS

Following is a list of integrals for reference in the solution of practical problems. The deduction of these integrals will be a useful exercise in the review of the earlier part of the book.

GENERAL FORMULÆ OF INTEGRATION

1. $\int (u \pm v \pm w \pm \cdots)\, dx = \int u\, dx \pm \int v\, dx \pm \int w\, dx \pm \cdots.$

2. $\int mu\, dx = m \int u\, dx.$

3 (*a*). $\int u\, dv = uv - \int v\, du.$

3 (*b*). $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx.$

ALGEBRAIC FORMS

4. $\int \frac{dx}{x} = \log x.$

5. $\int x^n\, dx = \frac{x^{n+1}}{n+1}$, when n is different from -1.

Expressions containing Integral Powers of $a + bx$

6. $\int \frac{dx}{a + bx} = \frac{1}{b} \log (a + bx).$

7. $\int (a + bx)^n\, dx = \frac{(a + bx)^{n+1}}{b(n+1)}$, when n is different from -1.

8. $\int F(x,\ a+bx)\,dx$. Try one of the substitutions, $z = a + bx$, $xz = a + bx$.

9. $\displaystyle\int\frac{x\,dx}{a+bx} = \frac{1}{b^2}[a+bx-a\log(a+bx)].$

10. $\displaystyle\int\frac{x^2\,dx}{a+bx} = \frac{1}{b^3}[\tfrac{1}{2}(a+bx)^2-2a(a+bx)+a^2\log(a+bx)].$

11. $\displaystyle\int\frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}.$

12. $\displaystyle\int\frac{dx}{x^2(a+bx)} = -\frac{1}{ax}+\frac{b}{a^2}\log\frac{a+bx}{x}.$

13. $\displaystyle\int\frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx)+\frac{a}{a+bx}\right].$

14. $\displaystyle\int\frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx)-\frac{a^2}{a+bx}\right].$

15. $\displaystyle\int\frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)}-\frac{1}{a^2}\log\frac{a+bx}{x}.$

16. $\displaystyle\int\frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx}+\frac{a}{2(a+bx)^2}\right].$

EXPRESSIONS CONTAINING a^2+x^2, a^2-x^2, $a+bx^n$, $a+bx^2$

17. $\displaystyle\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$; $\displaystyle\int\frac{dx}{1+x^2} = \tan^{-1}x.$

18. $\displaystyle\int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$; $\displaystyle\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}.$

19. $\displaystyle\int\frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{b}{a}}$, when $a>0$ and $b>0$.

20. $\displaystyle\int\frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\log\frac{a+bx}{a-bx}.$

21. $\int x^m (a+bx^n)^p\,dx$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.$$

22. $\int x^m (a+bx^n)^p\,dx$

$$=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}+\frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}\,dx.$$

23. $\int\frac{dx}{x^m(a+bx^n)^p}$

$$=-\frac{1}{(m-1)ax^{m-1}(a+bx^n)^{p-1}}-\frac{(m-n+np-1)b}{(m-1)a}\int\frac{dx}{x^{m-n}(a+bx^n)^p}.$$

24. $\int\frac{dx}{x^m(a+bx^n)^p}$

$$=\frac{1}{an(p-1)x^{m-1}(a+bx^n)^{p-1}}+\frac{m-n+np-1}{an(p-1)}\int\frac{dx}{x^m(a+bx^n)^{p-1}}.$$

25. $\int\frac{(a+bx^n)^p\,dx}{x^m}$

$$=-\frac{(a+bx^n)^{p+1}}{a(m-1)x^{m-1}}-\frac{b(m-n-np-1)}{a(m-1)}\int\frac{(a+bx^n)^p\,dx}{x^{m-n}}.$$

26. $\int\frac{(a+bx^n)^p\,dx}{x^m}$

$$=\frac{(a+bx^n)^p}{(np-m+1)x^{m-1}}+\frac{anp}{np-m+1}\int\frac{(a+bx^n)^{p-1}\,dx}{x^m}.$$

27. $\int\frac{x^m\,dx}{(a+bx^n)^p}$

$$=\frac{x^{m-n+1}}{b(m-np+1)(a+bx^n)^{p-1}}-\frac{a(m-n+1)}{b(m-np+1)}\int\frac{x^{m-n}\,dx}{(a+bx^n)^p}.$$

28. $\int\frac{x^m\,dx}{(a+bx^n)^p}$

$$=\frac{x^{m+1}}{an(p-1)(a+bx^n)^{p-1}}-\frac{m+n-np+1}{an(p-1)}\int\frac{x^m\,dx}{(a+bx^n)^{p-1}}.$$

29. $$\int\frac{dx}{(a^2+x^2)^n}$$
$$=\frac{1}{2(n-1)a^2}\left[\frac{x}{(a^2+x^2)^{n-1}}+(2n-3)\int\frac{dx}{(a^2+x^2)^{n-1}}\right].$$

30. $$\int\frac{dx}{(a+bx^2)^n}$$
$$=\frac{1}{2(n-1)a}\left[\frac{x}{(a+bx^2)^{n-1}}+(2n-3)\int\frac{dx}{(a+bx^2)^{n-1}}\right].$$

31. $$\int\frac{x\,dx}{(a+bx^2)^n}=\frac{1}{2}\int\frac{dz}{(a+bz)^n},\ \text{where } z=x^2.$$

32. $$\int\frac{x^2\,dx}{(a+bx^2)^n}$$
$$=\frac{-x}{2b(n-1)(a+bx^2)^{n-1}}+\frac{1}{2b(n-1)}\int\frac{dx}{(a+bx^2)^{n-1}}.$$

33. $$\int\frac{dx}{x(a+bx^n)}=\frac{1}{an}\log\frac{x^n}{a+bx^n}.$$

34. $$\int\frac{dx}{x^2(a+bx^2)^n}=\frac{1}{a}\int\frac{dx}{x^2(a+bx^2)^{n-1}}-\frac{b}{a}\int\frac{dx}{(a+bx^2)^n}.$$

35. $$\int\frac{x\,dx}{a+bx^2}=\frac{1}{2b}\log\left(x^2+\frac{a}{b}\right).$$

36. $$\int\frac{x^2\,dx}{a+bx^2}=\frac{x}{b}-\frac{a}{b}\int\frac{dx}{a+bx^2}.$$

37. $$\int\frac{dx}{x(a+bx^2)}=\frac{1}{2a}\log\frac{x^2}{a+bx^2}.$$

38. $$\int\frac{dx}{x^2(a+bx^2)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{dx}{a+bx^2}.$$

39. $$\int\frac{dx}{(a+bx^2)^2}=\frac{x}{2a(a+bx^2)}+\frac{1}{2a}\int\frac{dx}{a+bx^2}.$$

Expressions containing $\sqrt{a+bx}$

[See Formulæ 21–28]

40. $$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$$

41. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$$

42. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

43. $$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$$

44. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}, \text{ for } a>0.$$

45. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}, \text{ for } a<0.$$

46. $$\int \frac{dx}{x^2\sqrt{a+bx}} = \frac{-\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$$

47. $$\int \frac{\sqrt{a+bx}\,dx}{x} = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}.$$

Expressions containing $\sqrt{x^2+a^2}$

[See Formulæ 21–28]

48. $$\int (x^2+a^2)^{\frac{1}{2}}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right).$$

49. $$\int (x^2+a^2)^{\frac{3}{2}}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^4}{8}\log\left(x+\sqrt{x^2+a^2}\right).$$

50. $$\int (x^2+a^2)^{\frac{n}{2}}\,dx = \frac{x(x^2+a^2)^{\frac{n}{2}}}{n+1} + \frac{na^2}{n+1}\int (x^2+a^2)^{\frac{n}{2}-1}\,dx.$$

51. $$\int x(x^2+a^2)^{\frac{n}{2}}dx=\frac{(x^2+a^2)^{\frac{n+2}{2}}}{n+2}.$$

52. $$\int x^2(x^2+a^2)^{\frac{1}{2}}dx=\frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\log(x+\sqrt{x^2+a^2}).$$

53. $$\int\frac{dx}{(x^2+a^2)^{\frac{1}{2}}}=\log(x+\sqrt{x^2+a^2}).$$

54. $$\int\frac{dx}{(x^2+a^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{x^2+a^2}}.$$

55. $$\int\frac{x\,dx}{(x^2+a^2)^{\frac{1}{2}}}=\sqrt{x^2+a^2}.$$

56. $$\int\frac{x^2\,dx}{(x^2+a^2)^{\frac{1}{2}}}=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\log(x+\sqrt{x^2+a^2}).$$

57. $$\int\frac{x^2\,dx}{(x^2+a^2)^{\frac{3}{2}}}=-\frac{x}{\sqrt{x^2+a^2}}+\log(x+\sqrt{x^2+a^2}).$$

58. $$\int\frac{dx}{x(x^2+a^2)^{\frac{1}{2}}}=\frac{1}{a}\log\frac{x}{a+\sqrt{x^2+a^2}}.$$

59. $$\int\frac{dx}{x^2(x^2+a^2)^{\frac{1}{2}}}=-\frac{\sqrt{x^2+a^2}}{a^2x}.$$

60. $$\int\frac{dx}{x^3(x^2+a^2)^{\frac{1}{2}}}=-\frac{\sqrt{x^2+a^2}}{2a^2x^2}+\frac{1}{2a^3}\log\frac{a+\sqrt{x^2+a^2}}{x}.$$

61. $$\int\frac{(x^2+a^2)^{\frac{1}{2}}dx}{x}=\sqrt{a^2+x^2}-a\log\frac{a+\sqrt{a^2+x^2}}{x}.$$

62. $$\int\frac{(x^2+a^2)^{\frac{1}{2}}dx}{x^2}=-\frac{\sqrt{x^2+a^2}}{x}+\log(x+\sqrt{x^2+a^2}).$$

EXPRESSIONS CONTAINING $\sqrt{x^2 - a^2}$

[See Formulæ 21–28]

63. $$\int (x^2 - a^2)^{\frac{1}{2}}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right).$$

64. $$\int (x^2 - a^2)^{\frac{3}{2}}\, dx = \frac{x}{8}(2\,x^2 - 5\,a^2)\sqrt{x^2 - a^2} + \frac{3\,a^4}{8}\log\left(x + \sqrt{x^2 - a^2}\right).$$

65. $$\int (x^2 - a^2)^{\frac{n}{2}}\, dx = \frac{x\,(x^2 - a^2)^{\frac{n}{2}}}{n+1} - \frac{na^2}{n+1}\int (x^2 + a^2)^{\frac{n}{2}-1}\, dx.$$

66. $$\int x\,(x^2 - a^2)^{\frac{n}{2}}\, dx = \frac{(x^2 - a^2)^{\frac{n+2}{2}}}{n+2}.$$

67. $$\int x^2\,(x^2 - a^2)^{\frac{1}{2}}\, dx = \frac{x}{8}(2\,x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 - a^2}\right).$$

68. $$\int \frac{dx}{(x^2 - a^2)^{\frac{1}{2}}} = \log\left(x + \sqrt{x^2 - a^2}\right).$$

69. $$\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}} = -\frac{x}{a^2\sqrt{x^2 - a^2}}.$$

70. $$\int \frac{x\,dx}{(x^2 - a^2)^{\frac{1}{2}}} = \sqrt{x^2 - a^2}.$$

71. $$\int \frac{x^2\,dx}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right).$$

72. $$\int \frac{x^2\,dx}{(x^2 - a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2 - a^2}} + \log\left(x + \sqrt{x^2 - a^2}\right).$$

73. $$\int \frac{dx}{x\,(x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{a}\sec^{-1}\frac{x}{a}; \quad \int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x.$$

74. $$\int \frac{dx}{x^2\,(x^2 - a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2 - a^2}}{a^2 x}.$$

75. $$\int\frac{dx}{x^3(x^2-a^2)^{\frac{1}{2}}}=\frac{\sqrt{x^2-a^2}}{2\,a^2x^2}+\frac{1}{2\,a^3}\sec^{-1}\frac{x}{a}.$$

76. $$\int\frac{(x^2-a^2)^{\frac{1}{2}}\,dx}{x}=\sqrt{x^2-a^2}-a\cos^{-1}\frac{a}{x}.$$

77. $$\int\frac{(x^2-a^2)^{\frac{1}{2}}\,dx}{x^2}=-\frac{\sqrt{x^2-a^2}}{x}+\log\left(x+\sqrt{x^2-a^2}\right).$$

EXPRESSIONS CONTAINING $\sqrt{a^2-x^2}$

[See Formulæ 21–28]

78. $$\int(a^2-x^2)^{\frac{1}{2}}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

$$\int(a^2-x^2)^{\frac{3}{2}}\,dx=\frac{x}{8}(5\,a^2-2\,x^2)\sqrt{a^2-x^2}+\frac{3\,a^4}{8}\sin^{-1}\frac{x}{a}.$$

80. $$\int(a^2-x^2)^{\frac{n}{2}}\,dx=\frac{x\,(a^2-x^2)^{\frac{n}{2}}}{n+1}+\frac{a^2n}{n+1}\int(a^2-x^2)^{\frac{n}{2}-1}\,dx.$$

81. $$\int x\,(a^2-x^2)^{\frac{n}{2}}\,dx=-\frac{(a^2-x^2)^{\frac{n+2}{2}}}{n+2}.$$

82. $$\int x^2\,(a^2-x^2)^{\frac{1}{2}}\,dx=\frac{x}{8}(2\,x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}.$$

83. $$\int\frac{dx}{(a^2-x^2)^{\frac{1}{2}}}=\sin^{-1}\frac{x}{a};\quad\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x.$$

84. $$\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{a^2-x^2}}.$$

85. $$\int\frac{x\,dx}{(a^2-x^2)^{\frac{1}{2}}}=-\sqrt{a^2-x^2}.$$

86. $$\int\frac{x^2\,dx}{(a^2-x^2)^{\frac{1}{2}}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

87. $$\int\frac{x^2\,dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\frac{x}{a}.$$

88. $$\int \frac{x^m\,dx}{(a^2-x^2)^{\frac{1}{2}}} = -\frac{x^{m-1}}{m}\sqrt{a^2-x^2} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}}{(a^2-x^2)^{\frac{1}{2}}}\,dx.$$

89. $$\int \frac{dx}{x(a^2-x^2)^{\frac{1}{2}}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}}.$$

90. $$\int \frac{dx}{x^2(a^2-x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$$

91. $$\int \frac{dx}{x^3(a^2-x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2-x^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\log\frac{x}{a+\sqrt{a^2-x^2}}.$$

92. $$\int \frac{(a^2-x^2)^{\frac{1}{2}}}{x}\,dx = \sqrt{a^2-x^2} - a\log\frac{a+\sqrt{a^2-x^2}}{x}.$$

93. $$\int \frac{(a^2-x^2)^{\frac{1}{2}}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}.$$

Expressions containing $\sqrt{2\,ax-x^2}$, $\sqrt{2\,ax+x^2}$

[See Formulæ 21-28]

94. $$\int \sqrt{2\,ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2\,ax-x^2} + \frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}.$$

95. $$\int \frac{dx}{\sqrt{2\,ax-x^2}} = \operatorname{vers}^{-1}\frac{x}{a};\quad \int \frac{dx}{\sqrt{2\,x-x^2}} = \operatorname{vers}^{-1}x.$$

96. $$\int x^m\sqrt{2\,ax-x^2}\,dx = -\frac{x^{m-1}(2\,ax-x^2)^{\frac{3}{2}}}{m+2}$$
$$+\frac{(2\,m+1)a}{m+2}\int x^{m-1}\sqrt{2\,ax-x^2}\,dx.$$

97. $$\int \frac{dx}{x^m\sqrt{2\,ax-x^2}} = -\frac{\sqrt{2\,ax-x^2}}{(2\,m-1)\,ax^m}$$
$$+\frac{m-1}{(2\,m-1)\,a}\int \frac{dx}{x^{m-1}\sqrt{2\,ax-x^2}}.$$

98. $$\int \frac{x^m\,dx}{\sqrt{2\,ax-x^2}} = -\frac{x^{m-1}\sqrt{2\,ax-x^2}}{m} + \frac{(2\,m-1)\,a}{m}\int \frac{x^{m-1}dx}{\sqrt{2\,ax-x^2}}.$$

99. $$\int \frac{\sqrt{2\,ax-x^2}}{x^m}\,dx = -\frac{(2\,ax-x^2)^{\frac{3}{2}}}{(2\,m-3)ax^m} + \frac{m-3}{(2\,m-3)a}\int \frac{\sqrt{2\,ax-x^2}}{x^{m-1}}\,dx.$$

100. $$\int x\sqrt{2\,ax-x^2}\,dx = -\frac{3\,a^2+ax-2\,x^2}{6}\sqrt{2\,ax-x^2} + \frac{a^3}{2}\,\text{vers}^{-1}\frac{x}{a}.$$

101. $$\int \frac{dx}{x\sqrt{2\,ax-x^2}} = -\frac{\sqrt{2\,ax-x^2}}{ax}.$$

102. $$\int \frac{x\,dx}{\sqrt{2\,ax-x^2}} = -\sqrt{2\,ax-x^2} + a\,\text{vers}^{-1}\frac{x}{a}.$$

103. $$\int \frac{x^2\,dx}{\sqrt{2\,ax-x^2}} = -\frac{x+3\,a}{2}\sqrt{2\,ax-x^2} + \frac{3}{2}a^2\,\text{vers}^{-1}\frac{x}{a}.$$

104. $$\int \frac{\sqrt{2\,ax-x^2}}{x}\,dx = \sqrt{2\,ax-x^2} + a\,\text{vers}^{-1}\frac{x}{a}.$$

105. $$\int \frac{\sqrt{2\,ax-x^2}}{x^2}\,dx = -\frac{2\sqrt{2\,ax-x^2}}{x} - \text{vers}^{-1}\frac{x}{a}.$$

106. $$\int \frac{\sqrt{2\,ax-x^2}}{x^3}\,dx = -\frac{(2\,ax-x^2)^{\frac{3}{2}}}{3\,ax^3}.$$

107. $$\int \frac{dx}{(2\,ax-x^2)^{\frac{3}{2}}} = \frac{x-a}{a^2\sqrt{2\,ax-x^2}}.$$

108. $$\int \frac{x\,dx}{(2\,ax-x^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{2\,ax-x^2}}.$$

109. $$\int F(x,\ \sqrt{2\,ax-x^2})\,dx = \int F(z+a,\ \sqrt{a^2-z^2})\,dz, \text{ where } z=x-a.$$

110. $$\int \frac{dx}{\sqrt{2\,ax+x^2}} = \log\left(x+a+\sqrt{2\,ax+x^2}\right).$$

111. $$\int F(x,\ \sqrt{2\,ax+x^2})\,dx = \int F(z-a,\ \sqrt{z^2-a^2})\,dz, \text{ where } z=x+a.$$

EXPRESSIONS CONTAINING $a + bx \pm cx^2$

112. $$\int \frac{dx}{a + bx + cx^2} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{2cx + b}{\sqrt{4ac - b^2}}, \text{ when } b^2 < 4ac.$$

113. $$= \frac{1}{\sqrt{b^2 - 4ac}} \log \frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}}, \text{ when } b^2 > 4ac.$$

114. $$\int \frac{dx}{a + bx - cx^2} = \frac{1}{\sqrt{b^2 + 4ac}} \log \frac{\sqrt{b^2 + 4ac} + 2cx - b}{\sqrt{b^2 + 4ac} - 2cx + b}.$$

115. $$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log (2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}).$$

116. $$\int \sqrt{a + bx + cx^2}\, dx$$
$$= \frac{2cx + b}{4c}\sqrt{a + bx + cx^2} - \frac{b^2 - 4ac}{8c^{\frac{3}{2}}} \log (2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}).$$

117. $$\int \frac{dx}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2cx - b}{\sqrt{b^2 + 4ac}}.$$

118. $$\int \sqrt{a + bx - cx^2}\, dx$$
$$= \frac{2cx - b}{4c}\sqrt{a + bx - cx^2} + \frac{b^2 + 4ac}{8c^{\frac{3}{2}}} \sin^{-1} \frac{2cx - b}{\sqrt{b^2 + 4ac}}.$$

119. $$\int \frac{x\, dx}{\sqrt{a + bx + cx^2}}$$
$$= \frac{\sqrt{a + bx + cx^2}}{c} - \frac{b}{2c^{\frac{3}{2}}} \log (2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}).$$

120. $$\int \frac{x\, dx}{\sqrt{a + bx - cx^2}} = -\frac{\sqrt{a + bx - cx^2}}{c} + \frac{b}{2c^{\frac{3}{2}}} \sin^{-1} \frac{2cx - b}{\sqrt{b^2 + 4ac}}.$$

OTHER ALGEBRAIC EXPRESSIONS

121. $$\int \sqrt{\frac{a + x}{b + x}}\, dx = \sqrt{(a + x)(b + x)} + (a - b) \log (\sqrt{a + x} + \sqrt{b + x}).$$

122. $$\int \sqrt{\frac{a - x}{b + x}}\, dx = \sqrt{(a - x)(b + x)} + (a + b) \sin^{-1} \sqrt{\frac{x + b}{a + b}}.$$

123. $\int\sqrt{\frac{a+x}{b-x}}dx = -\sqrt{(a+x)(b-x)}-(a+b)\sin^{-1}\sqrt{\frac{b-x}{a+b}}.$

124. $\int\sqrt{\frac{1+x}{1-x}}dx = -\sqrt{1-x^2}+\sin^{-1}x.$

125. $\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}.$

EXPONENTIAL AND TRIGONOMETRIC EXPRESSIONS

126. $\int a^x dx = \frac{a^x}{\log a}.$

127. $\int e^x dx = e^x.$

128. $\int e^{ax} dx = \frac{e^{ax}}{a}.$

129. $\int \sin x\, dx = -\cos x.$

130. $\int \cos x\, dx = \sin x.$

131. $\int \tan x\, dx = \log \sec x = -\log \cos x.$

132. $\int \cot x\, dx = \log \sin x.$

133. $\int \sec x\, dx = \int\frac{dx}{\cos x} = \log(\sec x + \tan x) = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$

134. $\int \operatorname{cosec} x\, dx = \int\frac{dx}{\sin x} = \log(\operatorname{cosec} x - \cot x) = \log\tan\frac{x}{2}.$

135. $\int \sec^2 x\, dx = \tan x.$

136. $\int \operatorname{cosec}^2 x\, dx = -\cot x.$

137. $\int \sec x \tan x\, dx = \sec x.$

138. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x.$

139. $\int \sin^2 x\, dx = \frac{x}{2}-\frac{1}{4}\sin 2x.$

140. $\displaystyle\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x.$

141. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

142. $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

143. $\displaystyle\int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\,\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}.$

144. $\displaystyle\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\,\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}.$

145. $\displaystyle\int \cos^m x\sin^n x\,dx = \frac{\cos^{m-1} x\sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x\sin^n x\,dx.$

146. $\displaystyle\int \cos^m x\sin^n x\,dx$

$$= -\frac{\sin^{n-1} x\cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x\sin^{n-2} x\,dx.$$

147. $\displaystyle\int \frac{dx}{\sin^m x\cos^n x}$

$$= \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1} x\cos^{n-1} x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x\cos^{n-2} x}.$$

148. $\displaystyle\int \frac{dx}{\sin^m x\cos^n x}$

$$= -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1} x\cos^{n-1} x} + \frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2} x\cos^n x}.$$

149. $\displaystyle\int \frac{\cos^m x\,dx}{\sin^n x} = -\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x\,dx}{\sin^{n-2} x}.$

150. $\displaystyle\int \frac{\cos^m x\,dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} x\,dx}{\sin^n x}.$

151. $\displaystyle\int \sin x\cos^n x\,dx = -\frac{\cos^{n+1} x}{n+1}.$

152. $\displaystyle\int \sin^n x\cos x\,dx = \frac{\sin^{n+1} x}{n+1}.$

153. $$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$

154. $$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx.$$

155. $$\int \sin mx \sin nx\,dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}.$$

156. $$\int \cos mx \cos nx\,dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}.$$

157. $$\int \sin mx \cos nx\,dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}.$$

158. $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right), \text{ when } a>b.$$

159. $$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}, \text{ when } a<b.$$

160. $$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}, \text{ when } a>b.$$

161. $$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}, \text{ when } a<b.$$

162. $$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right).$$

163. $$\int e^{ax}\sin nx\,dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2+n^2};$$

$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}.$$

164. $$\int e^{ax}\cos nx\,dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2+n^2};$$

$$\int e^x \cos x\,dx = \frac{e^x(\sin x + \cos x)}{2}.$$

ANSWERS TO THE EXAMPLES

CHAPTER II

Art. 12

3. $y^a = ce^x$; $y^a = e^x$.

4. $y^2 = k(x^2 + c^2)$.

CHAPTER III

Art. 18

4. $\frac{x^5}{5}$, $\frac{x^{11}}{11}$, $\frac{s^{m+n+1}}{m+n+1}$, $\frac{t^{100}}{100}$, $\frac{z^{22}}{22}$, $\frac{p^{38}}{38}$.

5. $\frac{8}{225}$, $\frac{3}{40}$, $-\frac{1}{6x^6}$, $-\frac{1}{9x^9}$, $-\frac{1}{mx^m}$, $-\frac{1}{99x^{99}}$, $-\frac{1}{20x^{20}}$.

6. $\frac{3}{5}x^{\frac{5}{3}}$, $\frac{5}{6}t^{\frac{6}{5}}$, $\frac{4}{7}x^{\frac{7}{4}}$, $\frac{q}{p+q}v^{\frac{p}{q}+1}$, $\frac{7}{5}x^{\frac{5}{7}}$, $\frac{5}{2}x^{\frac{2}{5}}$, $\frac{q}{q-p}x^{-\frac{p}{q}+1}$.

7. $\frac{2}{3}x^{\frac{3}{2}}$, $2x^{\frac{1}{2}}$, $-\frac{2}{\sqrt{x}}$, $\frac{2}{5}x^{\frac{5}{2}}$, $\frac{5}{7}x^{\frac{7}{5}}$, $\frac{2}{7}x^{\frac{7}{2}}$, $4x^{\frac{1}{4}}$.

8. $\log t$, $\log(s-1)$, $\frac{4}{3}\log(x^3+3)$, $\log(uvw+1)$, $\log(x^3+2x^2-2x+4)$, $\log\tan x$, $\log\sin x$.

9. e^{2x}, $\frac{15}{\log 2}$, $\frac{(m+n)^x}{\log(m+n)}$.

10. $-\cos 2x$, $\sin 3x$, $\tan 4x$.

11. $\sec\frac{1}{2}x$, $\sin^{-1}2x$, $\sin^{-1}3x$, $\sin\ {}^{1}uv$.

12. $\tan^{-1}2x$, $\tan^{-1}3x$, $\sec^{-1}2x$, $\sec^{-1}4x$, $\sec^{-1}\frac{x}{a}$.

Art. 19

2. $\frac{7}{5}x^5$, $\frac{az^{11}}{11}$, $\frac{ac^2bx^d}{d}$, $\frac{a^2x^{\sqrt{2}+1}}{\sqrt{2}+1}$, $2b^{\frac{3}{2}}cx^{2a\sqrt{b}}$.

3. $\frac{1}{3}x^3 - x^2 + 5x$, $\frac{1}{4}x^4 - 2x^{\frac{3}{2}} + 2x^{\frac{5}{2}}$.

4. $9t + 15t^2 + \frac{25}{3}t^3 + c$, or $\frac{1}{15}(3 + 5t)^3 + c$, $a^2x - \frac{9}{5}a^{\frac{4}{3}}x^{\frac{5}{3}} + \frac{9}{7}a^{\frac{2}{3}}x^{\frac{7}{3}} - \frac{1}{3}x^3$, $\sin\theta - \cos\theta$.

5. $\frac{e^{ax}}{a}$, $\frac{e^{2x}}{2}$, $\frac{2}{3}e^{\frac{3}{2}x}$, $\frac{b}{p}e^{\frac{p}{b}x}$, $\frac{1}{2}e^{x^2+4x+3}$.

6. $\frac{\sin mx}{m} - \frac{\cos 3x}{3}$, $\frac{\tan 2x}{2} - \frac{\cot(m+n)x}{m+n}$.

7. $\frac{1}{nb}\log(a + bx^n)$, $\frac{1}{9}\log(4 + 3v^3)$, $-\frac{1}{8}\log(5 - 2x^4)$.

8. $\frac{a^2}{4}\sin^{-1}\theta$, $\frac{1}{6}\tan^{-1}\frac{2x}{3}$, $\frac{1}{20}\tan^{-1}\frac{5y}{4}$.

Art. 20

8. $\frac{2}{5}(x+a)^{\frac{5}{2}}$, $\frac{2}{3}(x+a)^{\frac{3}{2}}$, $\log(x+a)$, $2\sqrt{x+a}$, $-\frac{1}{x+a}$, $\frac{5}{18}(2+3x)^{\frac{6}{5}}$, $-\frac{3}{35}(3-7x)^{\frac{5}{3}}$.

9. $\sin(x+a)$, $\tan(x+a)$, $-\frac{1}{3}\tan(4-3x)$, $-\frac{1}{b}\cos(a+bx)$.

10. $2\sin\frac{x}{2}$, $\frac{1}{5}e^{2+5x}$, $-3e^{-\frac{x}{3}}$, $-\frac{1}{2\sin^2 x}$.

11. $\frac{3}{28}(4x - 3a)(a+x)^{\frac{4}{3}}$, $\frac{3}{10b^2}(a+bx)^{\frac{2}{3}}(2bx - 3a)$.

12. $\log\tan^{-1}x$, $\sin\log x$, $-n\cot\frac{\theta}{n}$.

13. $\frac{1}{4b}(a+bz)^4$, $\frac{3}{5b}(a+bx)^{\frac{5}{3}}$, $\frac{7}{3b}(a+by)^{\frac{3}{7}}$.

14. $\frac{1}{2}\sin^{-1}\frac{2x-1}{4}$.

15. $-\frac{1}{2}\operatorname{cosec}^2 x$, $\frac{1}{5}\sin^5\theta - \frac{3}{4}\sin^4\theta + \frac{4}{3}\sin^3\theta + \frac{11}{2}\sin^2\theta + 2\sin\theta$, $\frac{1}{4}\tan^4\phi - \frac{7}{3}\tan^3\phi + \tan^2\phi + 9\tan\phi$.

Art. 21

5. $x\sin^{-1}x + \sqrt{1-x^2}$.

6. $x\cot^{-1}x + \frac{1}{2}\log(1+x^2)$.

7. $a^z\left\{\frac{z}{\log a} - \frac{1}{(\log a)^2}\right\}$.

8. $a^x\left\{\frac{x^2}{\log a} - \frac{2x}{(\log a)^2} + \frac{2}{(\log a)^3}\right\}$.

9. $x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$.

10. $2\cos x + 2x\sin x - x^2\cos x$.

11. $x^2\sin x + 2x\cos x - 2\sin x$.

12. $\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2}$.

13. $x[(\log x)^2 - 2\log x + 2]$.

14. $\sin\theta(\log\sin\theta - 1)$.

15. $\tan x(\log\tan x - 1)$.

16. $\frac{x^4}{4}[(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}]$.

17. $\frac{1}{m^2}e^{mx}(mx - 1)$.

18. $\frac{x^{m+1}}{m+1}\left(\log x - \frac{1}{m+1}\right)$.

Art. 23

6. $\log(2x-5+2\sqrt{x^2-5x})$.

7. $\log(7x+\sqrt{7}\sqrt{7x^2+19})$.

8. $\frac{5}{2\sqrt{3}}\log(3x^2+1+\sqrt{3}\sqrt{3x^4+2x^2-1})$.

9. $\frac{1}{2}\log\frac{x-1}{x-3}$.

10. $5\sin^{-1}\left(\frac{x-2}{2}\right)$, or $5\operatorname{vers}^{-1}\frac{x}{2}$.

11. $\sin^{-1}\frac{x}{2}$.

12. $\sin^{-1}\frac{\theta}{\sqrt{5}}$.

13. $2\log\sec 3x$.

14. $\frac{7}{\sqrt{5}}\sin^{-1}\frac{\sqrt{5}x}{\sqrt{3}}$.

15. $\frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}$.

16. $\frac{1}{4\sqrt{2}}\log\frac{y-\sqrt{8}}{y+\sqrt{8}}$.

17. $\frac{1}{a}\log\sin ax$.

18. $\frac{1}{a}\log\sin(ax+b)$.

19. $\frac{1}{\sqrt{3}}\tan^{-1}\frac{2z}{\sqrt{3}}$.

20. $\frac{1}{2}\tan^{-1}\frac{x-2}{2}$.

21. $\frac{1}{4\sqrt{3}}\log\frac{x-2-2\sqrt{3}}{x-2+2\sqrt{3}}$.

22. $\frac{1}{2}\tan 2x+\log\tan\left(x+\frac{\pi}{4}\right)+x$.

23. $\frac{1}{b}\sec^{-1}\frac{x-a}{b}$.

24. $\frac{1}{4\sqrt{5}}\sec^{-1}\frac{2x}{\sqrt{5}}$.

25. $\frac{1}{2}\log(x^2+\sqrt{x^4-c^4})$.

26. $\frac{y}{3}(x^2-y^2)^{\frac{3}{2}}$.

27. $\log(\beta+\sqrt{\beta^2+2\sqrt{3}})$.

28. $\frac{1}{\sqrt{6}}\operatorname{vers}^{-1}\frac{4x}{5}$.

29. $\frac{1}{2}\tan^{-1}\frac{x+3}{2}$.

30. $\log\tan\frac{\theta}{2}+\log\sin\theta$.

31. $\frac{1}{a}\log(a^2x+b+a\sqrt{a^2x^2+2bx+c})$.

32. $\sqrt{x^2-1}+\log(x+\sqrt{x^2-1})$.

Art. 24

3. $\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}+c$.

4. $a^2x-\frac{y^3}{3}-xy^2-x^2y+c$.

5. $ax^2+bxy+ay^2+gx+ey+k$.

Page 55

1. acx^c, x^{m+n}, $\frac{(m+n)v^{m+n+2}}{m+n+2}$, $\left(\frac{a}{3}\right)^2x^{\frac{3}{a}}$, $b^{\frac{1}{5}}z^{\sqrt{ab}}$.

2. $\frac{1}{3}x^3+4x^{\frac{1}{2}}-\frac{9}{5}x^{\frac{5}{3}}+15x^{\frac{1}{3}}$, $7a^{\frac{4}{3}}-\frac{186}{5}a^{\frac{2}{3}}+\frac{381}{7}$.

3. $\frac{x^4}{4}-\frac{2x^3}{3}+2x^2-8x+25\log(x+2)$, $\frac{z^3}{3}+z^2+6z+5\log(z-2)$, $24+6\log 3$.

4. $\frac{n}{1+n}x^{\frac{1+n}{n}}$, $\frac{n}{n-1}x^{\frac{n-1}{n}}$, $\frac{1}{1-n}x^{1-n}$, $\frac{3}{5}v^{\frac{5}{3}}-\frac{9}{4}av^{\frac{4}{3}}+6av^{\frac{2}{3}}$.

5. $-2\cot 2\theta$, $\frac{3}{2}\sin 2\phi+\frac{5}{3}\cos 3\phi$, $-\frac{1}{b}\log(a+b\cos\psi)$.

6. $\log(y^2-a^2)$, $\sin x+\tan x$, $\frac{1}{2}(\log x)^2$, $\frac{[\log(ax+b)]^2}{2a}$.

7. 106, $\frac{1}{4}$, $\frac{1}{4}$, 4, $2n\pi+\frac{\pi}{2}$, $\log 2$. **8.** 2, $\frac{1}{2}$, $\frac{1}{a}+a$, $e^{\frac{1}{2}}-e^{-\frac{1}{2}}$.

9. $\frac{1}{4}e^5$, $\log 2$, $-\frac{(\log 2)^2}{8}$, $e^{\frac{\pi}{2}}-1$, $\frac{1}{a}\left(e^{\frac{a\pi}{3}}-e^{\frac{a\pi}{4}}\right)$.

10. $4\sqrt{a}$, $\frac{1}{12}(1+3\sqrt{3}-2\sqrt{2})$, $3+\sqrt{15}$.

12. $\sin^{-1}\frac{3+2x}{\sqrt{13}}$, $2\tan^{-1}\sqrt{1+x}$. **13.** $\frac{3}{8}x^{\frac{4}{3}}-\frac{1}{3}x^{\frac{3}{2}}$, $\frac{bx^2-2a}{3b^2}\sqrt{a+bx^2}$.

14. $x\sec^{-1}x-\log(x+\sqrt{x^2-1})$, $x\operatorname{cosec}^{-1}x+\log(x+\sqrt{x^2-1})$, $x\cos^{-1}x-\sqrt{1-x^2}$, $\frac{x^3}{3}\sin^{-1}x+\frac{1}{9}(x^2+2)\sqrt{1-x^2}$.

15. $x\sin x+\cos x$, $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x$, $x\tan x+\log\cos x-\frac{x^2}{2}$.

16. $a^x\left[\frac{x^3}{\log a}-\frac{3x^2}{(\log a)^2}+\frac{6x}{(\log a)^3}-\frac{6}{(\log a)^4}\right]$, $\cos x(1-\log\cos x)$, $\cot x(1-\log\cot x)$.

17. $(x^2-2x+5)\sin(x^2-2x+5)+\cos(x^2-2x+5)$.

18. $\frac{(x^3+a^3)^2}{6}[2\log(x^3+a^3)-1]-a^3(x^3+a^3)[\log(x^3+a^3)-1]$.

19. $-\frac{2}{3x^{\frac{3}{2}}}\left\{(\log x)^2+\frac{4}{3}\log x+\frac{8}{9}\right\}$. **22.** $\tan x-\cot x+2\log\tan x$.

23. $\frac{\theta}{2}-\frac{1}{2}\log(1+\cot\theta)+\frac{1}{4}\log(1+\cot^2\theta)$.

24. $\sin^{-1}\frac{x}{\sqrt{a^2+b^2}}$. **25.** 0. **28.** $\frac{1}{a}\sec^{-1}\frac{x}{a}-\frac{1}{x}\log(x+\sqrt{x^2-a^2})$.

26. $a\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)+b\log\sin\theta$. **29.** $\frac{1}{2}\{\log(x+\sqrt{x^2+a^2})\}^2$.

27. $\sin^{-1}\frac{x-1}{\sqrt{3}}$, $\sin^{-1}\frac{x+5}{3}$. **30.** $\log\sec\left(\frac{\theta}{2}+\frac{\pi}{4}\right)$.

31. $\log \dfrac{\sin^b \theta}{\cos^a \theta}$.

32. $\frac{2}{3} x^{\frac{3}{2}} \operatorname{vers}^{-1} \dfrac{x}{a} + \frac{4}{9}(x + 4\,a)\sqrt{2\,a - x}$.

33. $\dfrac{x}{2}\sqrt{x^2 \pm a^2} \pm \dfrac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2})$.

34. $\dfrac{1}{\sqrt{5}} \log \dfrac{\sqrt{x} - \sqrt{5}}{\sqrt{x} + \sqrt{5}}$.

35. $\dfrac{1}{\sqrt{2}} \log \tan \left(\dfrac{x}{2} + \dfrac{\pi}{8}\right)$.

36. $\frac{1}{2} \log \tan \left(\phi + \dfrac{\pi}{4}\right)$.

37. $14 \sin^{-1} \dfrac{x}{2} - (\frac{3}{2}\,x + 10)\sqrt{4 - x^2}$.

38. $\dfrac{1}{2\sqrt{3}} \log \dfrac{\sec x - \sqrt{3}}{\sec x + \sqrt{3}}$.

39. $\dfrac{1}{2\sqrt{5}} \log \dfrac{e^x - \sqrt{5}}{e^x + \sqrt{5}}$.

40. $\dfrac{1}{\sqrt{b}\,\log a} \sec^{-1} \dfrac{a^x}{\sqrt{b}}$.

41. $2\sqrt{2} \sin^{-1} \left(\sqrt{2} \sin \dfrac{x}{2}\right)$.

42. $\dfrac{2}{\sqrt{4\,ac - b^2}} \tan^{-1} \dfrac{2\,ax + b}{\sqrt{4\,ac - b^2}}$, when $b^2 < 4\,ac$,

$\dfrac{1}{\sqrt{b^2 - 4\,ac}} \log \dfrac{2\,ax + b - \sqrt{b^2 - 4\,ac}}{2\,ax + b + \sqrt{b^2 - 4\,ac}}$, when $b^2 > 4\,ac$.

43. $\dfrac{1}{\sqrt{a}} \log (2\,ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c})$.

44. $\dfrac{1}{\sqrt{a}} \sin^{-1} \dfrac{2\,ax - b}{\sqrt{b^2 + 4\,ac}}$.

45. $-\cos x \cos y + c$.

46. $\sin x \cos y + c$.

47. $x^3 + 3\,x^2 y + 4\,xy^2 + 2\,y^3 + c$.

CHAPTER IV

Art. 29

1. (*a*) $\frac{35}{2}$; (*b*) 4.
2. (*a*) $74\frac{2}{3}$; (*b*) $\frac{5}{3}$.
3. (*a*) $\frac{1}{18}$; (*b*) 4.
4. $9\frac{5}{6}$.
5. $28\frac{13}{15}$.
6. $\frac{1}{6}$.
7. $25\frac{3}{5}$.
8. $\frac{3}{5}\sqrt[3]{1024}$.
9. $2\frac{2}{3}$.
10. 24.
11. 2.
12. $\frac{1}{4}\,\pi r^2$.
13. $\dfrac{a^2}{6}$.
14. $k^2 \log \dfrac{b}{a}$.

Art. 30

5. $81\frac{1}{30}\,\pi$.
6. (*a*) $\frac{500}{3}\,\pi$; (*b*) $\frac{9}{28}\,\pi$; (*c*) $\frac{9}{100}\,\pi$; $\frac{125}{576}\,\pi$.
7. (*a*) $\frac{3}{2}\,\pi$; (*b*) $2\,\pi$; (*c*) $\frac{245}{3}\,\pi$; (*d*) $\frac{81}{8}\,\pi$.
8. (*a*) $4\,\pi$; (*b*) $\pi a^2 \log (1 + \sqrt{2})$.
9. $\frac{3}{5}\,\pi c^{\frac{2}{3}} x_1^{\frac{5}{3}}$.
10. $\dfrac{\pi}{7} c^{\frac{1}{3}} x_1^{\frac{7}{3}}$.
11. $\dfrac{\pi y_1^7}{7\,c^2}$.
12. $\frac{4}{3}\,\pi ab^2$.

Art. 32

3. $\sqrt{n}\,y = x + c$; $\sqrt{n}\,(y - 3) = x - 2$.

4. $y = ce^{kx}$.

5. $(n + 1)\,y^2 = 2\,kx^{n+1} + c$.

6. $cr = e^{\frac{\theta}{k}}$.

7. $r^n = c \sin n\theta$; $r = c \sin \theta$; $r = c\,(1 - \cos\theta)$.

Page 76

1. $-\frac{47}{60}\frac{a^5}{b^3}$.

2. $4\frac{2}{3}$.

3. $\frac{16}{3}a^2$.

4. $\frac{c^2}{2}\left(e - \frac{1}{e}\right)$.

6. $\log 4 - \frac{3}{4}$.

7. (a) $\frac{a^4}{12}$; (b) $\frac{7\,a^4}{12}$.

8. $\frac{1}{2}$.

9. $\frac{4}{3}a^3$.

10. $2\,a^2$.

11. $2\,\pi a^2$.

12. πa^2.

13. $\frac{32 \cdot b^{\frac{7}{2}}}{3 \cdot 5 \cdot 7 \cdot a^{\frac{3}{2}}}$.

14. $\frac{4}{3}ab$.

15. (a) $\frac{17}{12}\pi$; (b) $\pi^2 a^3$.

16. $\frac{32}{105}\pi a^3$.

17. $\frac{\pi a^3}{15}$.

18. $\frac{\pi a^3}{2}\left(e + \frac{5}{e} - 4\right)$.

19. $\frac{8\,\pi a^3}{3}(\log 8 - 2)$.

20. $2\,\pi^2 a^3$.

21. $\frac{301}{3}\pi$.

23. $350\,\pi^2$.

26. $\pi a^3\left(\frac{5}{3} - \frac{\pi}{2}\right)$.

CHAPTER V

Art. 34

2. $\log\frac{x - 3}{x + 2}$.

3. $\log\,(x + 5)^2(x - 7)^3$.

4. $-\log\,(x-2)^2(x-1)$.

5. $\log\frac{\sqrt{x^2 - 1}}{x}$.

6. $\log\frac{x^2 - 1}{x}$.

7. $\log\frac{(x - a)^a}{(x - b)^b}$.

8. $\log\,[\sqrt{2\,x - 1}\,(x+2)]$.

9. $\frac{1}{3}\log x(x^2 - 3)^4$.

10. $\log\,(x + 3)^2(x - 2)$.

11. $\log\frac{x}{1 - x^2}$.

12. $\log\frac{x - 2 - \sqrt{3}}{x - 2 + \sqrt{3}}$.

13. $x^2 + \log(x+1)^3(x-4)$.

14. $\frac{x^2}{2} + 5\,x - \log x^3(x+2)$.

15. $\log\frac{(x - p)(x + q)}{x}$.

16. $x + \log\frac{x - 2}{x - 1}$.

17. $\log\sqrt{x^2 + 2\,x - 4}$.

18. $\frac{1}{12}\log x(x - 1)^6(x - 2)^{-8}(x - 3)^9$.

19. $\frac{1}{2\sqrt{2}}\log\frac{x - \sqrt{2}}{x + \sqrt{2}} + \frac{1}{2\sqrt{3}}\log\frac{x - \sqrt{3}}{x + \sqrt{3}}$.

20. $\frac{1}{6}\log\,(x^2 - 1) - \frac{1}{8}\log x(x^2 - 2) + \frac{1}{48}\log\,(x^2 - 4)$.

21. $x + \frac{(\alpha - a)(\alpha - b)(\alpha - c)}{(\alpha - \beta)(\alpha - \gamma)}\log\,(x - \alpha) + \frac{(\beta - a)(\beta - b)(\beta - c)}{(\beta - \alpha)(\beta - \gamma)}\log\,(x - \beta)$
$+ \frac{(\gamma - a)(\gamma - b)(\gamma - c)}{(\gamma - \alpha)(\gamma - \beta)}\log\,(x - \gamma)$.

Art. 35

2. $\log(x+1)+\dfrac{1}{x+1}.$

3. $\log(x-3)-\dfrac{2}{x-3}.$

4. $\log\sqrt{2x+1}-\dfrac{3}{2}\cdot\dfrac{1}{2x+1}.$

5. $\log(x+a)^a(x+b)^{-b}-\dfrac{b^2}{x+b}.$

6. $\dfrac{1}{(3\sqrt{5}-2-x)^2}.$

7. $\log(x-3)^3-\dfrac{6}{x-3}+\dfrac{9}{2(x-3)^2}.$

8. $\log\dfrac{x+1}{x}-\dfrac{2}{x+1}.$

9. $a\log(x+a)+\dfrac{2a^2}{x+a}-\dfrac{a^3}{2(x+a)^2}.$

10. $\log x(x-1)^3-\dfrac{2}{x}.$

11. $\dfrac{7}{2x+2}+\dfrac{11}{4}\log\dfrac{x+1}{x+3}.$

12. $\dfrac{4}{x+2}+\log(x+1).$

13. $-\dfrac{x}{4(x^2-2)}+\dfrac{1}{8\sqrt{2}}\log\dfrac{x+\sqrt{2}}{x-\sqrt{2}}.$

14. $\dfrac{2x-5}{(x-2)^2}+\dfrac{2x+1}{2(x+1)^2}+\log\dfrac{x-2}{x+1}.$

15. $-\dfrac{x}{(x-1)^2}+\log\dfrac{(x-1)^2}{x}.$

Art. 36

3. $\log x+2\tan^{-1}x.$

4. $\log(x+1)^2+\tan^{-1}x.$

5. $\dfrac{x^3}{3}+\tfrac{1}{3}\log x+\dfrac{\sqrt{2}}{3}\tan^{-1}\dfrac{x}{\sqrt{2}}.$

6. $\tan^{-1}x+\displaystyle\int\frac{dx}{(x^2+1)^2}.$

7. $\tfrac{1}{3}\log(3x+2)-\tfrac{1}{2}\tan^{-1}(x+1).$

8. $3\log x+\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x-3}{\sqrt{2}}.$

9. $\dfrac{x^3}{3}-\dfrac{7x^2}{2}+5x.$

10. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}}+\displaystyle\int\frac{dx}{(x^2+3)^2}+\int\frac{dx}{(x^2+3)^3}.$

11. $\dfrac{1}{\sqrt{5}}\tan^{-1}\dfrac{x}{\sqrt{5}}-\dfrac{1}{6}\cdot\dfrac{3x^2+16}{(x^2+5)^2}.$

12. $x+\dfrac{1}{6}\log\dfrac{x^2+3}{x^2}-\sqrt{3}\tan^{-1}\dfrac{x}{\sqrt{3}}.$

13. $\log\dfrac{x^2+4}{\sqrt{x^2+2}}+\tfrac{3}{2}\tan^{-1}\dfrac{x}{2}-\dfrac{3}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}.$

CHAPTER VI

Art. 38

2. $\dfrac{x}{a^2\sqrt{a^2-x^2}}.$

3. $-\dfrac{x}{a^2\sqrt{x^2-a^2}}.$

4. $\dfrac{x}{a^2\sqrt{x^2+a^2}}.$

5. $-\dfrac{\sqrt{a^2-x^2}}{a^2x}.$

6. $-\dfrac{\sqrt{x^2+a^2}}{a^2x}.$

7. $\dfrac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3}.$

8. $-\frac{1}{a}\log\left(\frac{a+\sqrt{a^2+x^2}}{x}\right).$

11. $-\frac{\sqrt{2\,ax-x^2}}{ax}.$

9. $-\frac{1}{a}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$

12. $\frac{x}{a\sqrt{2\,ax-x^2}}.$

10. $-\frac{1}{a}\sin^{-1}\frac{a}{x}.$

13. $\frac{-(2\,ax-x^2)^{\frac{3}{2}}}{3\,ax^3}.$

Art. 40

3. $3\,x^{\frac{1}{3}}+\tfrac{1}{2}\log\frac{(x^{\frac{1}{3}}-1)^3}{x-1}-\sqrt{3}\tan^{-1}\left(\frac{2\,x^{\frac{1}{3}}+1}{\sqrt{3}}\right).$

4. $x^{\frac{1}{3}}+\log(x^{\frac{1}{5}}+1).$

5. $14\left(x^{\frac{1}{14}}-\frac{x^{\frac{1}{7}}}{2}+\frac{x^{\frac{3}{14}}}{3}-\frac{x^{\frac{2}{7}}}{4}+\frac{x^{\frac{5}{14}}}{5}\right).$

6. $12\left(\frac{x^{\frac{3}{4}}}{9}-\frac{x^{\frac{2}{3}}}{4}-\frac{x^{\frac{7}{12}}}{7}+\frac{x^{\frac{1}{2}}}{3}+\frac{x^{\frac{5}{12}}}{5}-\frac{x^{\frac{1}{3}}}{4}-\frac{x^{\frac{1}{4}}}{3}+x^{\frac{1}{6}}+x^{\frac{1}{12}}-\log(x^{\frac{1}{6}}+1)-\tan^{-1}x^{\frac{1}{12}}\right).$

7. $12\left(\frac{x^{\frac{5}{12}}}{5}-\frac{7\,x^{\frac{1}{3}}}{4}+\frac{x^{\frac{1}{4}}}{3}+\frac{5\,x^{\frac{1}{6}}}{2}+x^{\frac{1}{12}}+\log(x^{\frac{1}{12}}-1)^3(x^{\frac{1}{12}}+1)^2\right).$

8. $\frac{2}{15\,b^2}(a+bx)^{\frac{3}{2}}(3\,bx-2\,a).$

9. $\log(3+2\sqrt{x+1}).$

10. $x+1+4\sqrt{x+1}+4\log(\sqrt{x+1}-1).$

11. $\tfrac{3}{40}(c-x)^{\frac{5}{3}}(5\,x+3\,c-24).$

12. $\frac{3}{8}\,\frac{2\,x+9}{(2\,x+3)^{\frac{1}{3}}}.$

Art. 41

3. $-\frac{x^2+2}{3}\sqrt{1-x^2}.$

4. $\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{1-x^2}-\sqrt{2}}{\sqrt{1-x^2}+\sqrt{2}}\right).$

5. $\sqrt{x^2+5}-\frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{x^2+5}-\sqrt{3}}{\sqrt{x^2+5}+\sqrt{3}}\right).$

Art. 43

2. $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}\,x}\right).$

3. $\frac{1}{\sqrt{2}}\log(3+4\,x+2\sqrt{2}\sqrt{2\,x^2+3\,x+4}).$

4. $\log(x+1+\sqrt{2\,x+x^2})-\frac{4}{x+\sqrt{2\,x+x^2}}.$

5. $\sqrt{x^2+x+1}+\tfrac{1}{2}\log(x+\tfrac{1}{2}+\sqrt{x^2+x+1}).$

6. $-\frac{\sqrt{1-x^2}}{2\,x^2}-\tfrac{1}{2}\log\left(\frac{\sqrt{1-x^2}+1}{x}\right).$

7. $-2\sqrt{\frac{6-x}{x}}+\cos^{-1}\left(\frac{x-3}{3}\right).$

Art. 44

2. $\sqrt{5}\log(5x-1+\sqrt{5}\sqrt{5x^2-2x+7})$.

3. $\frac{1}{\sqrt{3}}\log(6x-1+2\sqrt{3}\sqrt{3x^2-x+1})$.

4. $\sin^{-1}\left(\frac{x-1}{2}\right)$.

5. $2\sqrt{2}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)$.

6. $-\frac{3}{2}\sqrt{6-3x-2x^2}+\frac{11}{4\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{57}}\right)$.

7. $a\sin^{-1}\frac{x}{a}-\sqrt{a^2-x^2}$.

8. $5\sqrt{x^2-3}+\log(x+\sqrt{x^2-3})$.

9. $\frac{7}{3}\sqrt{3x^2-3x+1}+\frac{13}{2\sqrt{3}}\log(6x-3+2\sqrt{3}\sqrt{3x^2-3x+1})$.

12. $\frac{1}{5}\sqrt{5x^2-26x+34}+\frac{33}{5\sqrt{5}}\log(5x-13+\sqrt{5}\sqrt{5x^2-26x+34})$
$$+13\log\left(\frac{2x-7+\sqrt{5x^2-26x+34}}{x-3}\right).$$

13. $-\log\left(\frac{2+x+2\sqrt{x^2+x+1}}{x}\right)$.

14. $2\sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right)-4\sqrt{2}\log\left(\frac{\sqrt{2}+\sqrt{1+2x-x^2}}{x-1}\right)$.

Art. 45

4. $-\sqrt{2ax-x^2}+a\operatorname{vers}^{-1}\frac{x}{a}$.

5. $-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$.

6. $\frac{x}{a^2\sqrt{a^2-x^2}}$.

7. $-\frac{\sqrt{a^2-x^2}}{2a^2x^2}-\frac{1}{2a^3}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$.

8. $-\frac{1}{6}(3a^2+ax-2x^2)\sqrt{2ax-x^2}+\frac{a^3}{2}\operatorname{vers}^{-1}\frac{x}{a}$.

9. $\frac{-(a^2+2x^2)\sqrt{a^2-x^2}}{3a^4x^3}$.

10. $\frac{1}{3}(x^2-2a^2)\sqrt{x^2+a^2}$.

Page 98

1. $\log(2x+a+2\sqrt{x^2+ax})$.

2. $\log(2x-5+2\sqrt{x^2-5x+6})$.

3. $\frac{M}{N}\log[2N^2x-P+2N\sqrt{N^2x^2-Px+R}]$.

4. $\log(x-a)(2x-a-b+2\sqrt{(x-a)(x-b)})$.

5. $\frac{1}{3\sqrt{2}}\log(4x^3+3+2\sqrt{2}\sqrt{2x^6+3x^3+1})$.

6. $\frac{1}{\sqrt{3}}\sin^{-1}\sqrt{3(x+1)}$. 7. $\frac{1}{\sqrt{5}}\log[\sqrt{5}(x+1)+\sqrt{5x^2+10x-27}]$.

8. $-\frac{1}{\sqrt{m}}\sin^{-1}\frac{mx}{n}$. 9. $\frac{x}{17}+\frac{g}{17}\sin^{-1}(x-1)+\frac{h}{17}\frac{x-1}{\sqrt{2x-x^2}}$.

10. $\frac{1}{2}\log(2x+\sqrt{4x^2-9})+\frac{1}{\sqrt{6}}\sin^{-1}\left(\frac{12x-13}{5}\right)$.

11. $-2\sqrt{x^2+2x+4}+6\log(x+1+\sqrt{x^2+2x+4})$.

12. $-\frac{1}{4}(2x+11\sqrt{(x-2)(3-x)}+\frac{15}{8}\sin^{-1}(2x-5)$

13. $\frac{1}{2}(x-4)\sqrt{x^2-1}+\frac{3}{2}\log(x+\sqrt{x^2-1})$.

14. $-\frac{1}{\sqrt{2}}\log\left(\frac{1-x+\sqrt{2}\sqrt{x^2+1}}{x+1}\right)$. 15. $-\log\left(\frac{\sqrt{x^2-2}-3x-4}{2x+3}\right)$.

16. $\log(x+2+\sqrt{x^2+4x+5})+\frac{7}{\sqrt{2}}\log\left(\frac{\sqrt{2}\sqrt{x^2+4x+5}-x-1}{x+3}\right)$.

17. $\log\left(\frac{x+1+\sqrt{1+2x-x^2}}{x}\right)-\frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}+\sqrt{1+2x-x^2}}{x-1}\right)$.

18. $\tan^{-1}\sqrt{x^2-3}$.

19. $\sqrt{x^2+1}+2\log(x+\sqrt{x^2+1})-\frac{3}{\sqrt{2}}\log\left(\frac{1-x+\sqrt{2}\sqrt{x^2+1}}{x+1}\right)$.

20. $\frac{3}{2}\sqrt{\frac{1+x}{1-x}}+\sin^{-1}x+\frac{3}{4\sqrt{2}}\sin^{-1}\left(\frac{3x+1}{x+3}\right)$.

21. $\frac{a^3}{2}x^4+ab^2x^2+\frac{1}{4}\sqrt{b^2+a^2x^2}(3b^2x+2a^2x^3)+\frac{b^4}{4a}\log(\sqrt{b^2+a^2x^2}+ax)$.

22. $\frac{x(3a^2-2x^2)}{3a^4(a^2-x^2)^{\frac{3}{2}}}$.

23. $-\frac{x}{48}\sqrt{a^2-x^2}(8x^4+10a^2x^2+15a^4)+\frac{5a^6}{16}\sin^{-1}\frac{x}{a}$.

24. $-\sqrt{a^2-x^2}$. 25. $-\frac{1}{15}\sqrt{a^2-x^2}(3x^4+4a^2x^2+8a^4)$.

26. $-\frac{1}{a}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$. 27. $\sqrt{a^2-x^2}-a\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$.

23. $\frac{x}{8}\sqrt{a^2-x^2}(2x^2-a^2)+\frac{a^4}{8}\sin^{-1}\frac{x}{a}$.

29. $\frac{x}{48}\sqrt{a^2-x^2}(8x^4-2a^2x^2-3a^4)+\frac{a^6}{16}\sin^{-1}\frac{x}{a}$.

30. $\mp\frac{\sqrt{x^2\pm a^2}}{a^2x}$. 31. $-\frac{1}{a}\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$.

32. $\frac{x}{8}\sqrt{x^2\pm a^2}(2x^2\pm a^2)-\frac{a^4}{8}\log(x+\sqrt{x^2\pm a^2})$. 33. $\frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$.

34. $\frac{x}{8}\sqrt{x^2 \pm a^2}\,(2\,x^2 \pm 5\,a^2) + \frac{3\,a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).$ 35. $\sqrt{x^2 \pm a^2}.$

36. $-\sqrt{2\,ax - x^2} + a\,\text{vers}^{-1}\frac{x}{a}.$ 37. $-\frac{\sqrt{2\,ax - x^2}}{ax}.$

38. $-\frac{2\,x^2 + 5\,a\,(x + 3\,a)}{6}\sqrt{2\,ax - x^2} + \frac{5\,a^3}{2}\,\text{vers}^{-1}\frac{x}{a}.$

39. $\frac{x - a}{2}\sqrt{2\,ax - x^2} + \frac{a^2}{2}\,\text{vers}^{-1}\frac{x}{a}.$ 40. $\frac{x\,(3\,x^2 + 5\,a^2)}{8\,a^4\,(a^2 + x^2)^2} + \frac{3}{8\,a^5}\tan^{-1}\frac{x}{a}.$

41. $\frac{x^2}{4\,a^4\,(x^4 + a^4)} + \frac{1}{4\,a^6}\tan^{-1}\frac{x^2}{a^2}.$ 42. $-\frac{1}{3\,(a^3 - x^3)}.$

43. $\frac{x - 1}{4\,(x^2 - 2\,x + 3)} + \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x - 1}{\sqrt{2}}\right).$ 44. $\frac{x}{2\,(1 - x^2)} + \frac{1}{4}\log\frac{x + 1}{x - 1}.$

45. $\sqrt{1 + x + x^2} - \frac{1}{2}\log\left(2\,x + 1 + 2\sqrt{1 + x + x^2}\right) - \log\left(\frac{1 - x + 2\sqrt{1 + x + x^2}}{x + 1}\right).$

46. $-\tan^{-1}\sqrt{x} + 2\sqrt{2}\tan^{-1}\sqrt{\frac{x}{2}} - \sqrt{3}\tan^{-1}\sqrt{\frac{x}{3}}.$

47. $\sin^{-1}\frac{2\,a^2b^2 - (a^2 + b^2)(a^2 + b^2 - x^2)}{(a^2 - b^2)(a^2 + b^2 - x^2)}.$ 48. $\frac{\pi^2 a^2}{4}.$

49. $\frac{2\,x - 1}{2\,(x^2 + 3)}.$ 50. $\frac{x\sqrt{x^2 + 2}}{2} + 4\log\left(x + \sqrt{x^2 + 2}\right).$

51. $\frac{2\,x + 5}{16}\sqrt{4\,x^2 + 4\,x + 3} + \frac{7}{8}\log\left(2\,x + 1 + \sqrt{4\,x^2 + 4\,x + 3}\right).$

52. $\frac{x - 1}{2}\sqrt{x^2 + 2\,x + 3}.$ 53. $-\frac{x}{18}\sqrt{-3 + 12\,x - 9\,x^2} - \frac{19}{18}\sin^{-1}(3\,x - 2).$

54. $\frac{1}{4\,l}(2\,lx + m)\sqrt{lx^2 + mx + n} + \frac{m^2 - 4\,ln}{8\,l\sqrt{-l}}\sin^{-1}\left(\frac{2\,lx + m}{\sqrt{m^2 - 4\,ln}}\right).$

55. $\frac{1}{2\,a^2\sqrt{2}}\log\left[\frac{(x + \sqrt{x^2 - a^2})^2 + (3 - 2\sqrt{2})\,a^2}{(x + \sqrt{x^2 - a^2})^2 + (3 + 2\sqrt{2})\,a^2}\right].$

56. $-\frac{2\sqrt{3\,x^2 - 2\,x + 1}}{2\,x - 1}.$

CHAPTER VII

Art. 46

5. (*a*) $\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x$;
   (*b*) $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$;
   (*c*) $-\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x.$

6. (*a*) $\frac{-\cos x\sin x}{4}(\sin^2 x+\frac{3}{2})+\frac{3}{8}x.$

(*b*) $-\frac{1}{6}\sin^5 x\cos x+\frac{5}{6}\int\sin^4 x\,dx,$ [see (*a*)];

(*c*) $-\frac{\sin^7 x\cos x}{7}+\frac{7}{8}\int\sin^6 x\,dx,$ [see (*b*)].

7. (*a*) $\frac{\sin x\cos x}{8}(2\cos^2 x+3)+\frac{3}{8}x.$

(*b*) $\frac{1}{6}\sin x\cos^5 x+\frac{5}{6}\int\cos^4 x\,dx,$ [see (*a*)].

(*c*) $\frac{1}{8}\sin x\cos^7 x+\frac{7}{8}\int\cos^6 x\,dx,$ [see (*b*)].

8. (*a*) $-\frac{1}{4}\frac{\cos x}{\sin^4 x}+\frac{3}{4}\int\frac{1}{\sin^3 x},$ (see Ex. 4).

(*b*) $\frac{1}{2}\sec x\tan x+\log\sqrt{\sec x+\tan x}.$

(*c*) $\frac{1}{3}\frac{\sin x}{\cos^3 x}+\frac{2}{3}\tan x.$

(*d*) $\frac{1}{4}\frac{\sin x}{\cos^4 x}+\frac{3}{4}\int\frac{dx}{\cos^3 x},$ [see (*b*)].

(*e*) $\frac{1}{5}\frac{\sin x}{\cos^5 x}+\frac{4}{5}\int\frac{dx}{\cos^4 x},$ [see (*c*)].

(*f*) $-\frac{1}{3}\frac{\cos x}{\sin^3 x}-\frac{2}{3}\cot x+c.$

(*g*) $-\frac{1}{5}\frac{\cos x}{\sin^5 x}+\frac{4}{5}\int\frac{dx}{\sin^4 x},$ [see (*f*)].

Art. 48

4. (*a*) $\frac{1}{3}\tan^3 x+\tan x+c.$

(*b*) $-\frac{1}{3}\cot^3 x-\cot x.$

(*c*) $-\frac{1}{5}\cot^5 x-\frac{2}{3}\cot^3 x-\cot x.$

(*d*) $\tan^3\frac{x}{3}+3\tan\frac{x}{3}+c.$

5. (*a*) $\frac{1}{2}\tan x\sec x+\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$

(*b*) $\frac{1}{4}\tan x\sec^3 x+\frac{3}{8}\left\{\tan x\sec x+\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}.$

(*c*) $-\frac{\cot x\operatorname{cosec} x}{2}+\frac{1}{2}\log\tan\frac{x}{2}.$

(*d*) $-\frac{1}{4}\cot x\operatorname{cosec}^3 x-\frac{3}{8}\left(\cot x\operatorname{cosec} x-\log\tan\frac{x}{2}\right).$

(*e*) $\frac{3}{4}\tan\frac{2}{3}x\sec\frac{2}{3}x+\frac{3}{4}\log\tan\left(\frac{\pi}{4}+\frac{x}{3}\right).$

Art. 49

2. (*a*) $\frac{1}{3}\tan^3 x-\tan x+x.$

(*b*) $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\log\sec x.$

(*c*) $\frac{1}{5}\tan^5 x-\frac{1}{3}\tan^3 x+\tan x-x.$

(*d*) $-\frac{1}{2}\cot^2 x-\log\sin x.$

(*e*) $-\frac{1}{3}\cot^3 x+\cot x+x.$

(*f*) $-\frac{1}{4}\cot^4 x+\frac{1}{2}\cot^2 x+\log\sin x.$

Art. 50

2. $\frac{3}{5}\sin^{\frac{5}{3}} x - \frac{6}{11}\sin^{\frac{11}{3}} x + \frac{3}{17}\sin^{\frac{17}{3}} x.$

3. $-\frac{4}{7}\cos^{\frac{7}{4}} x + \frac{4}{15}\cos^{\frac{15}{4}} x.$

4. $-2\sqrt{\cos x}\,(1 - \frac{2}{5}\cos^2 x + \frac{1}{9}\cos^4 x).$

5. $\frac{3}{2}\sin^{\frac{2}{3}} x\,(1 - \frac{1}{2}\sin^2 x + \frac{1}{7}\sin^4 x).$

6. $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + c.$

7. $-\frac{1}{7}\cot^7 x - \frac{1}{5}\cot^5 x + c.$

Art. 51

2. $\frac{\cos x}{2}\left(\frac{\sin^5 x}{3} - \frac{\sin^3 x}{12} - \frac{\sin x}{8}\right) + \frac{x}{16}.$

3. $\tan x - 2\cot x - \frac{1}{3}\cot^3 x.$

4. $-\frac{\cos x}{2\sin x}(3 - \cos^2 x) - \frac{3x}{2}.$

5. $-\frac{\cos x}{2\sin^2 x} - \cos x - \frac{3}{2}\log\tan\frac{x}{2}.$

Art. 52

2. $\frac{1}{2}\tan^2 x + 2\log\tan x - \frac{1}{2}\cot^2 x.$

3. $\frac{3}{11}\tan^{\frac{11}{3}} x + \frac{3}{5}\tan^{\frac{5}{3}} x.$

4. $-\frac{1}{6}\cot^6 x - \frac{1}{4}\cot^4 x.$

5. $-\frac{\operatorname{cosec}^9 x}{9} + \frac{2}{7}\operatorname{cosec}^7 x - \frac{1}{5}\operatorname{cosec}^5 x.$

6. $\frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x.$

7. $4\sec^{\frac{3}{4}} x\,(\frac{1}{19}\sec^2 x - \frac{2}{11}\sec x + \frac{1}{3}).$

Art. 53

2. $\frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right).$

3. $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x.$

4. $\frac{1}{16}(5x - 4\sin 2x + \frac{1}{3}\sin^3 2x + \frac{3}{4}\sin 4x).$

5. $\frac{1}{16}(5x + 4\sin 2x - \frac{1}{3}\sin^3 2x + \frac{3}{4}\sin 4x).$

6. $-\frac{\sin^3 2x}{48} + \frac{x}{16} - \frac{\sin 4x}{64}.$

7. $\frac{1}{128}\left(3x - \sin 4x + \frac{\sin 8x}{8}\right).$

Art. 55

1. $\frac{2}{\sqrt{5}}\tan^{-1}\frac{(3\tan\frac{x}{2} - 2)}{\sqrt{5}}.$

2. $\frac{1}{2\sqrt{5}}\log\frac{2\tan x + 3 - \sqrt{5}}{2\tan x + 3 + \sqrt{5}}.$

3. $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right).$

4. $\frac{1}{2}\tan^{-1}\left(2\tan\frac{x}{2}\right).$

5. $\frac{1}{6}\log\frac{\tan x + 3}{\tan x - 3}.$

6. $\frac{1}{6}\log\frac{2\tan x + 1}{\tan x + 2}.$

Art. 56

1. $\frac{1}{2}e^x(\sin x - \cos x).$

2. $\frac{1}{2}e^x(\sin x + \cos x).$

3. $\frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x).$

4. $\frac{1}{13}e^{-3x}(2\sin 2x - 3\cos 2x).$

5. $-\frac{1}{2}e^{-x}(\sin x + \cos x).$

6. $\frac{1}{2}e^x\left(1 + \frac{2\sin 2x + \cos 2x}{5}\right).$

Art. 57

1. $-\frac{\cos 8x}{16}-\frac{\cos 2x}{4}$.

2. $\frac{\sin 11x}{22}+\frac{\sin 3x}{6}$.

3. $-\frac{\sin 11x}{22}+\frac{\sin x}{2}$.

4. $\frac{3}{26}\sin\frac{13}{3}x+\frac{3}{10}\sin\frac{5}{3}x$.

5. $-\frac{1}{2}\cos x-\cos\frac{1}{2}x$.

6. $-\frac{1}{2}\sin x+\frac{5}{4}\sin\frac{2}{5}x$.

CHAPTER VIII

Art. 59

4. 240.

5. 80.

6. The double infinity of straight lines $y=c_1x+c_2$, in which c_1, c_2 are arbitrary constants.

7. $3y=2x(x^2-1)$.

8. $\frac{1}{5}(n^5-m^5)(d-c)(b-a)$.

9. $\pi(\beta-\alpha)$.

Art. 60

3. $24\frac{1}{2}$.

4. $58\frac{1}{2}$.

5. $\frac{143}{30}a^3$.

6. $\frac{4}{3}a^3$.

7. $6b^3$.

8. $32a^7$.

Art. 62

4. $\frac{(A^2-a^2)(B^2-b^2)}{4c}$.

5. $\frac{2}{3}a^3\tan\alpha$.

6. $\frac{abc}{90}$.

7. $\frac{4}{3}(\pi-\frac{4}{3})a^3$.

CHAPTER IX

Art. 65

2. $cr=e^{\frac{\theta}{k}}$.

3. $r^n=c\sin n\theta$; $r=c\sin\theta$; $r=c(1-\cos\theta)$.

Art. 66

3. $\frac{1}{6}a^2(\beta^3-\alpha^3)$.

4. $\frac{8a^2}{3}$.

5. $\frac{3}{8}\pi a^2$.

6. $\frac{3a^2}{2}$.

Art. 69

3. πab.

4. $5\pi^2a^3$.

Art. 70

3. $2\pi^2a^2b$.

4. $\frac{1}{3}Bh$.

5. $\frac{1}{2}\pi a^2h$.

6. $4\frac{1}{2}$ cubic feet.

7. $(\pi-\frac{4}{3})a^2h$.

8. $\pi\sqrt{pq}\,h^2$.

Art. 71

2. $s = \dfrac{y_1\sqrt{y_1^2 + 4a^2}}{4a} + a\log\dfrac{y_1 + \sqrt{y_1^2 + 4a^2}}{2a}$;

$= \sqrt{ax_1 + x_1^2} + a\log\dfrac{\sqrt{x_1} + \sqrt{a + x_1}}{\sqrt{a}}.$

$s = 2.29558\,a.$

3. $s = \dfrac{8a}{27}\left\{\left(1 + \dfrac{9x_1}{4a}\right)^{\frac{3}{2}} - 1\right\}$; $s = \frac{335}{27}a.$

4. $s = \dfrac{a}{2}\left(e^{\frac{x_1}{a}} - e^{-\frac{x_1}{a}}\right)$; $s = \dfrac{a}{2}\left(e - \dfrac{1}{e}\right).$

5. $s = 4a\left(\cos\dfrac{\theta_0}{2} - \cos\dfrac{\theta_1}{2}\right)$; length of a complete arch $= 8a$.

6. $s = 6a.$

Art. 72

3. $8a.$

4. $\dfrac{a}{2}\left[\theta_2\sqrt{1 + \theta_2^2} - \theta_1\sqrt{1 + \theta_1^2} + \log\dfrac{\theta_2 + \sqrt{1 + \theta_2}}{\theta_1 + \sqrt{1 + \theta_1}}\right].$

5. $\sqrt{a^2 + r_1^2} - \sqrt{a^2 + r_2^2} + a\log\left\{\dfrac{r_1(a + \sqrt{a^2 + r_2^2})}{r_2(a + \sqrt{a^2 + r_1^2})}\right\}.$

6. $\dfrac{r_1 - 1}{a}\sqrt{a^2 + 1}.$

7. $2a\left\{\sqrt{5} - 2 - \sqrt{3}\log\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{2}(2 + \sqrt{3})}\right\}.$

8. $s = a\tan\dfrac{\theta_1}{2}\sec\dfrac{\theta_1}{2} + a\log\tan\left(\dfrac{\theta_1}{4} + \dfrac{\pi}{4}\right)$; $s = 2a\left(\sec\dfrac{\pi}{4} + \log\tan\tfrac{3}{8}\pi\right).$

Art. 73

1. $s = r\phi.$

2. $s = 4a\left(1 - \cos\dfrac{\phi}{3}\right).$

3. (1) $s = 4a(1 - \cos\phi)$; (2) $s = 4a\sin\phi.$

4. (1) $s = p\tan\phi\sec\phi + p\log\tan\left(\dfrac{\phi}{2} + \dfrac{\pi}{4}\right)$;

(2) $s = p\tan\left(\phi + \dfrac{\pi}{4}\right)\sec\left(\phi + \dfrac{\pi}{4}\right) + p\log\tan\left(\dfrac{\phi}{2} + \dfrac{3\pi}{8}\right) - p\sqrt{2} - p\log 2p\sqrt{2}.$

5. $9s = 4a(\sec^3\phi - 1).$

6. $s = a\log\tan\left(\dfrac{\phi}{2} + \dfrac{\pi}{4}\right).$

7. $s = a(e^{c\phi} - 1).$

8. $s = c\log\sec\phi.$

9. $s = \dfrac{3a}{2}\sin^2\phi.$

Art. 74

2. $2\pi b\left(b + \frac{a^2}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b}{a}\right)$, or
$2\pi ab\left(\sqrt{1-e^2} + \frac{\sin^{-1}e}{e}\right)$,
in which e is the eccentricity.

3. $\frac{8}{3}\pi a^{\frac{1}{2}}(a+x)^{\frac{3}{2}}$.
4. $2\pi a^2\left(1-\frac{1}{e}\right)$.
5. $\frac{12}{5}\pi a^2$.
6. $\pi(\pi-2)a^2$.
7. $\frac{32}{5}\pi a^2$.

Art. 75

3. $2(\pi-2)a^2$.
4. $\frac{2\pi a}{\sqrt{AB}}$.

Art. 76

3. $26\frac{1}{2}$.
4. $-\frac{1}{4}$.
5. (a) $9\frac{5}{6}$; (b) $12\frac{1}{3}$; (c) $-\frac{47}{60}\frac{a^4}{b^2}$.
6. $.7854\,a$.
7. $.6366\,a$.
8. $.6366$.
9. $1.273\,a$.
10. $\frac{\pi}{2} - 1^c$, or 32.704°.
11. $\frac{1}{6}n^2$.

Art. 77

2. $\frac{a}{2}$.
3. $\frac{2}{3}a$.
4. $\frac{2}{3}a^2$.
5. $\frac{6}{5}a$.
6. $\frac{3}{4}a$.

Page 164

1. $\frac{4}{3}\pi a^3$.
2. Volume $= \frac{4}{3}\pi a^2 b$; surface $= 2\pi a^2 + \frac{\pi b^2}{e}\log\frac{1+e}{1-e}$, in which e is the eccentricity.
3. $\pi\left\{\left(x+\frac{a}{2}\right)\sqrt{ax+x^2} - \frac{a^2}{4}\log\frac{2x+a+2\sqrt{ax+x^2}}{a}\right\}$.
4. $4\pi^2a^3$.
5. $\pi b\sqrt{a^2+b^2}$.
6. $4\pi^2ab$.
7. $\frac{4}{3}ab^2\cot\alpha$.
8. $\frac{4}{3}a^2h$.
9. Volume $= \frac{16}{3}a^3$; surface $= 8a^2$.
10. $\frac{3}{2}\pi a^3$.
11. $\frac{4}{35}\pi a^3$.
12. Volume $= 2\pi a^3\left(\frac{2\sin\alpha}{3} + \frac{\sin\alpha\cos^2\alpha}{3} - \alpha\cos\alpha\right)$;
surface $= 4\pi a^2(\sin\alpha - \alpha\cos\alpha)$.
13. Volume $= \pi^2a^3$; surface $= \frac{32}{3}\pi a^2$.
14. Surface $= \frac{64}{3}\pi a^2$.
15. Volume $= \pi a^3\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)$; surface $= 8\pi a^2(\pi - \frac{4}{3})$.
16. $\frac{\pi ab^2}{6}(10-3\pi)$.
17. $\frac{1}{15}\pi(3a^2+4ab+8b^2)h$.

18. $2a\sqrt{3}$. **19.** $s=\sqrt{b^2+y^2}+b\log\dfrac{\sqrt{b^2+y^2}-b}{y}$, where $b=\dfrac{1}{\log a}$.

20. $\dfrac{8b(a+b)}{a}$. **21.** (a) $\dfrac{2(8-\sqrt{27})}{3\sqrt{3}}p$. (b) $2(\sqrt{27}-1)p$.

22. $\log\sqrt{3}$. **23.** $\dfrac{aa^2}{2}$. **24.** $\dfrac{3\pi a}{2}$. **25.** $\dfrac{m}{3}[(\theta_1^2+4)^{\frac{3}{2}}-8]$.

CHAPTER X

Art. 79

5. The density at a point three fourths of the distance from the vertex to the base, namely, $\frac{3}{4}kh$.

6. $\bar{x}=\frac{3}{4}h$. **7.** $\bar{x}=\frac{3}{4}h$; $x=\frac{2}{3}h$.

8. $\bar{x}=\bar{y}=\frac{2}{5}a$. **9.** $\bar{x}=5$. **11.** $\bar{x}=\dfrac{4a}{3\pi}$, $\bar{y}=0$.

12. Mass $=\frac{2}{3}ka^3$, if density $=k$ distance.
Mean density $=.4244$ max. density.
Center of mass is at $\bar{y}=0$, $\bar{x}=\frac{3}{16}\pi a=.589a$.

13. $\bar{y}=0$, $\bar{x}=\dfrac{2}{3}\dfrac{a\sin\alpha}{\alpha}$. **14.** $\bar{x}=\dfrac{4a}{3\pi}$, $\bar{y}=\dfrac{4b}{3\pi}$.

15. (a) $\bar{x}=\frac{3}{5}h$, $\bar{y}=0$;
(b) $\bar{x}=\frac{3}{5}h$, $\bar{y}=\frac{3}{8}k$, in which k is the ordinate corresponding to $x=h$.

17. $\bar{x}=\bar{y}=\frac{256}{315}\cdot\dfrac{a}{\pi}$. **18.** $\bar{x}=\bar{y}=\frac{1}{5}a$. **19.** $\bar{x}=\frac{5}{6}a$, $\bar{y}=0$.

20. $\bar{x}=-\frac{5}{3}a$, $\bar{y}=0$. **24.** At a point distant $\frac{2}{5}a$ from the base.

21. $\bar{x}=\frac{2}{3}x$, $\bar{y}=0$. **25.** $\bar{x}=-\frac{8}{5}a$, $\bar{y}=0$.

Art. 80

(In the answers M denotes mass)

4. If a is the radius, $I=\frac{1}{2}Ma^2$, $k=\dfrac{a}{\sqrt{2}}$.

5. $I=\dfrac{ab(a^2+b^2)}{12}$. **6.** $k=\dfrac{a}{\sqrt{6}}$. **7.** $\dfrac{ab^3}{20}$.

8. (a) $I=\frac{1}{4}Mb^2$; (b) $I=\frac{1}{4}Ma^2$; (c) $I=\frac{1}{4}M(a^2+b^2)$.

9. $I=M\dfrac{a^2}{3}$. **10.** $I=\frac{1}{5}M(b^2+c^2)$. **11.** $I=\frac{2}{3}Ma^2$. **12.** $I=\frac{2}{5}Ma^2$.

13. $I=\dfrac{4k}{15}a^5=\frac{2}{5}Ma^2$, since $M=\frac{2}{3}ka^3$ by Ex. 12, Art. 79.

14. $I=\dfrac{MR^2}{2}$; $k=\dfrac{R}{\sqrt{2}}$. $I=m\pi R^2l\left(\dfrac{R^2}{2}+\dfrac{l^2}{3}\right)$ or $M\left(\dfrac{R^2}{2}+\dfrac{l^2}{3}\right)$.

CHAPTER XI

Art. 82

3. $x - \frac{1}{2} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot x^9}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5 \cdot x^{13}}{2 \cdot 4 \cdot 6 \cdot 13} + \cdots + c.$

4. $x + \frac{1}{2} \cdot \frac{x^6}{6} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^{11}}{11} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^{16}}{16} + \cdots + c.$

5. $\frac{3}{4} x^{\frac{4}{3}} (1 - \frac{1}{5} x^2 - \frac{1}{32} x^4 - \frac{1}{88} x^6 + \cdots) + c.$

6. $\frac{1}{3} x^3 - \frac{1}{10} x^5 - \frac{1}{56} x^7 - \frac{1}{144} x^9 + \cdots + c.$

7. $x^m \left(\frac{1}{m} + \frac{pc}{q} \frac{x^n}{m + n} + \frac{p(p - q)c^2}{1 \cdot 2 \cdot q^2} \frac{x^{2n}}{m + 2n} + \cdots \right) + c.$

8. $2\sqrt{\sin x} \left(1 + \frac{1}{2} \frac{\sin^2 x}{5} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\sin^4 x}{9} + \cdots \right) + c.$

9. $\log x + x + \frac{x^2}{1 \cdot 2^2} + \frac{x^3}{1 \cdot 2 \cdot 3^2} + \cdots + c.$

10. $\log x + \frac{mx}{1} + \frac{m^2x^2}{1 \cdot 2^2} + \frac{m^3x^3}{1 \cdot 2 \cdot 3^2} + \cdots + c.$

11. $x - \frac{x^3}{3\underline{|3}} + \frac{x^5}{5\underline{|5}} - \frac{x^7}{7\underline{|7}} + \cdots + c.$

Art. 83

2. $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \cdots + \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} \cdot \frac{x^{2n+1}}{2n + 1} + \cdots.$

5. $\log (1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$

   $\log (1 - x) = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots.$

6. $\log (x + \sqrt{1 + x^2}) = x - \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots.$

CHAPTER XIII

Art. 97

4. $y = x\frac{dy}{dx} + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3.$

6. $x^2\frac{d^2y}{dx^2} + 2y = 2x\frac{dy}{dx}.$

5. $(1 - x^2)\left(\frac{dy}{dx}\right)^2 + 1 = 0.$

7. $\frac{d^2y}{dx^2} + m^2y = 0.$

Art. 98

2. $y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = c.$

3. $\tan y = c(1 - e^x)^3.$

Art. 99

2. $xy^2 = c^2(x + 2y)$. **3.** $y = ce^{\frac{x^3}{3y^3}}$.

Art. 100

1. $a^2x - \frac{y^3}{3} - xy^2 - x^2y = c$. **2.** $x^3 - 6x^2y - 6xy^2 + y^3 = c$.

3. $x^2y^2 + 4x^3y - 4xy^3 + y^3 - xe^y + e^{2x}y + x^4 = c$.

Art. 101

2. $2a \log x + a \log y - y = c$. **3.** $x^2 - y^2 - 1 = cx$.

4. $x^2e^x + my^2 = cx^2$.

Art. 102

2. $y = (x + c)e^{-x}$. **4.** $y(x^2 + 1)^2 = \tan^{-1} x + c$.

3. $y = x^2(1 + ce^{\frac{1}{x}})$. **5.** $x^n y = ax + c$.

Art. 103

2. $7y^{-\frac{1}{3}} = cx^{\frac{2}{3}} - 3x^3$. **3.** $y^{\frac{1}{2}} = c(1 - x^2)^{\frac{1}{4}} - \frac{1 - x^2}{3}$.

4. $60y^3(x + 1)^2 = 10x^6 + 24x^5 + 15x^4 + c$.

Art. 104

2. $343(y + c)^3 = 27ax^7$. **3.** $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$.

Art. 105

2. $\log(p - x) = \frac{x}{p - x} + c$, with the given relation. **3.** $x = \log p^2 + 6p + c$.

Art. 106

1. $y = c - a\log(p - 1)$, $x = c + a \log \frac{p}{p - 1}$. **2.** $y - c = \sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1 - x}{x}}$.

Art. 107

3. $y = cx + \sin^{-1} c$. **4.** $y = cx + \frac{m}{c}$. **5.** $y^2 = cx^2 + 1 + c$.

Art. 108

3. The catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ or $\frac{y}{a} = \cosh^{-1} \frac{x}{a}$.

4. The envelope of the family of lines $y = cx + \frac{ac}{c - 1}$, namely the parabola $(x - y)^2 - 2a(x + y) + a^2 = 0$.

5. The envelope of the lines $y = cx + a\sqrt{1+c^2}$, namely the circle $x^2 + y^2 = a^2$.

6. The circles $x^2 + y^2 = 2\,cx$.

7. The circles $x^2 + y^2 = k^2$.

8. The circles which pass through the origin and have their centers on the y-axis.

9. The rectangular hyperbolas $x^2 - y^2 = c^2$ whose axes coincide in direction with the asymptotes of the former system.

10. $x^2 + y^2 = 2\,a^2 \log x + c$.

12. The system of circles $r = c' \sin \theta$ which pass through the origin and touch the initial line.

13. $r^n \cos n\theta = c^n$.

14. The confocal and coaxial parabolas $r = \dfrac{2\,c}{1 - \cos \theta}$.

15. The system of curves $r^n = c^n \sin n\theta$; $r^2 = c^2 \sin 2\,\theta$, a series of lemniscates having their axis inclined at an angle of 45° to that of the given system.

Art. 109

2. $x = -\dfrac{f}{n^2} \sin nt + At + B$.

3. $x = \frac{1}{2} gt^2 + At + B$.

4. $y = \frac{1}{2} x^2 (l - \frac{1}{3} x)$.

5. $y = xe^x - 3\,e^x + c_1 x^2 + c_2 x + c_3$.

6. $y = c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1} + \dfrac{\underline{\lvert m}\, x^{m+n}}{\underline{\lvert m+n}}$.

Art. 110

2. $t = \sqrt{\dfrac{a}{2\,\mu}} \left\{ \dfrac{a}{2} \left(\text{vers}^{-1} \dfrac{2\,x}{a} - \pi \right) - \sqrt{ax - x^2} \right\}$.

3. $ax = \log (y + \sqrt{y^2 + c_1}) + c_2$, or $y = c_1' e^{ax} + c_2' e^{-ax}$.

Art. 111

2. $e^{-ay} = c_1 x + c_2$.

3. $15\,y = 8\,(x + c_1)^{\frac{5}{2}} + c_2 x + c_3$.

Art. 112

1. $y = c_1 \log x + c_2$.

2. $y = c_1 \sin ax + c_2 \cos ax + c_3 x + c_4$.

3. $y^2 = x^2 + c_1 x + c_2$.

Art. 114

3. $x = c_1e^{\frac{3}{2}t} + c_2e^{-4t}$.

4. $y = c_1e^{-2x} + e^x(c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x)$.

5. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x}$.

6. $y = c_1e^{-2x} + c_2e^{4x} + c_3e^x$.

7. $y = c_1e^{ax} + c_2e^{-ax} + c_3 \sin(ax + \alpha)$.

Art. 115

2. $y = c_1 + e^{-x}(c_2 + c_3x)$.

3. $y = e^{-x}(c_1 + c_2x + c_3x^2) + c_4e^{4x}$.

Art. 116

3. $y = c_1x^{-1} + c_2x^{-2}$.

4. $y = x^2(c_1 + c_2 \log x)$.

5. $y = x^2[c_1 + c_2 \log x + c_3(\log x)^2]$.

Page 227

2. $y^2 = 2\,xy\dfrac{dy}{dx} + x^2$.

3. $\tan x \tan y = k^2$.

4. $x^2 + y^2 = 2\,a^2 \tan^{-1}\dfrac{y}{x} + c$.

5. $xy(x - y) = c$.

6. $ax^2 + bxy + cy^2 + gx + ey = k$.

7. $x = cye^{\frac{1}{xy}}$

8. $x^2 + \dfrac{e^x}{y} = c$.

9. $y = \dfrac{x}{1 - a} - \dfrac{1}{a} + cx^a$.

10. $y = \tan x - 1 + ce^{-\tan x}$.

11. $y = ax + cx\sqrt{1 - x^2}$.

12. $y^{-n+1} = ce^{(n-1)\sin x} + 2 \sin x + \dfrac{2}{n-1}$.

13. $\dfrac{1}{y^2} = x^2 + 1 + ce^{x^2}$.

14. $y = c,\ x + y = c,\ xy + x^2 + y^2 = c$.

15. $2\,y = cx^2 + \dfrac{a}{c}$.

16. $y^2 = 2\,cx + c^2$.

17. $e^y = ce^x + c^3$.

18. $(c_1x + c_2)^2 + a = c_1y^2$.

19. $y = c_1 + c_2x + c_3e^{ax} + c_4e^{-ax}$.

20. $y = c_2 - \sin^{-1} c_1e^{-x}$.

21. $y = c_1e^{-x} + c_2 + \frac{1}{2}\,e^x$.

22. $y = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos \dfrac{\sqrt{3}}{2}x + c_3 \sin \dfrac{\sqrt{3}}{2}x\right)$.

23. $y = c \sin(nx + \alpha)(ax + b)$.

24. $y = e^x(c_1 + c_2x) \sin x + e^x(c_3 + c_4x) \cos x$.

25. $y = c_1 + c_2x + e^x(c_3 + c_4x)$.

26. $y = x(c_1 \cos \log x + c_2 \sin \log x) + c_3x^{-1}$.

27. $y = (c_1 + c_2 \log x) \sin \log x + (c_3 + c_4 \log x) \cos \log x$.

28. $E\,Iy = \dfrac{P}{2}\left(\dfrac{lx^2}{4} - \dfrac{x^3}{6}\right) + c_1x + c_2$.

29. $E\,Iy = -\,\gamma b\dfrac{x^5}{120} + c_1x^3 + c_2x^2 + c_3x + c_4$.

INDEX

[The numbers refer to pages.]

Algebraic transformations, 103, 105, 107, 108, 113.
Amsler, 188.
Amsler's planimeter, 189.
Angles, use of multiple, 114.
Anti-derivative (anti-differential), 1, 5, 7, 11, 12, 14, 21, 25.
Applications to mechanics, 167.
Approximate integration, 177–189.
  rules for, 181–188, 235.
Areas, change of variable, 141, 142.
  derivation of integration formulæ for, 9, 27.
  oblique axes, 140.
  polar coördinates, double integration, 139.
  polar coördinates, single integration, 135.
  precautions in finding, 63.
  rectangular coördinates, 58.
  rectangular coördinates, double integration, 126.
  rules for approximate determination of, 181–188, 235–237.
  surfaces of revolution, 152–156.
  surfaces $z = f(x, y)$, 156–160.
Auxiliary equation, 223, 224.

Bernouilli, James and John, 2.
Bertrand, 237.
Boussinesq, 186.

Cajori, 2.
Cardioid, area, 138.
  center of mass, 173.
  intrinsic equation, 151.
  length, 149.
  orthogonal trajectories, 216.
  surface of revolution, 156.
Carpenter, 189.
Catenary, area, 76.
  intrinsic equation, 150.
  length, 147.
  surface of revolution, 156.
  volume of revolution, 77.
Center of mass, 168.
Change of variable, 41.
Circle, area, 61, 139, 140, 141.
  evolute of, 166.
  intrinsic equation, 151.
  length, 146, 149.
  orthogonal trajectories, 217.
Cissoid, center of mass, 173.
  length, 149.
  volume of revolution, 77.
Clairaut, 213.
  equation of, 213.
Complementary function, 222.
Cone, center of mass, 172.
  moment of inertia, 175.
  surface, 164.
  volume, 71, 73, 143.
  volume of frustum, 77.
Conoid, volume, 143.
Constant of integration, 22.
  geometrical meaning of, 23.
  two kinds, 25.
Convergent, 178.
Cotes, 236.
Curves, areas of, oblique axes, 140.
  areas of, polar coördinates, 135–140.
  areas of, rectangular coördinates, 58.
  derived, 29.
  derivation of equations of, 25, 26, 75, 134, 214–217.
  integral, 33, 190–200, 240–245.
  intrinsic equations of, 149.
  quadrature of, 58.
Cycloid, area, 141.
  intrinsic equation, 151.

Cycloid, length, 147.
  note on length, 144.
  volume of revolution, 142.
  volumes and surfaces of revolution, 165.
Cylinder, 143.
  moment of inertia, 176.

Density, 167.
Derived curves, 29.
Derivation of equations of curves, 25, 26, 75, 134, 214–217.
Derivation of fundamental formulæ, 48.
Differential equations, *see* Equation.
Differential, integration of total, 52.
Differentiation under integration sign, 190.
Durand, 186, 187, 190, 237.

Equations, homogeneous in x, y, 205.
  linear, constant coefficients, 223.
  linear, homogeneous, 225.
  linear of first order, 208.
  linear of nth order, 222.
  linear, properties of, 222.
  of order higher than first, 218–228.
  reducible to linear form, 209.
  resolvable into component equations, 210.
  solutions, general, 202.
  solutions, particular, 202.
  solvable for x, 212.
  solvable for y, 211.
  variables easily separable, 205.
Evolute, of circle, length, 166.
  of parabola, length, 165.
Expansion of functions in series, 179.
Exponential functions, 117.

Figures of curves, 246–248.
Fisher, Irving, 30.
Folium of Descartes, area, 138.
Formulæ, areas, 9, 27, 137, 139, 141.
  lengths, 145, 148.
  of approximate integration, 183, 185, 186, 187, 235–237.
  of integration, fundamental, 37, 47, 48.
  of integration, table of, 249–262.
  of integration, universal, 39, 44.
  of reduction, 46, 93, 94, 95, 101, 102, 106, 110, 231–234.
  surfaces, 154, 155, 157, 158.
  volumes, 70, 130, 143.
Fourier, 9.
Fractions, rational, 78–83, 229–231.
Functions, irrational, 84–99.
  trigonometric and exponential, 100–118.

Geometrical applications, 58–77, 126–166, 214–217.
  meaning of constant of integration, 23.
  principle, 7.
  representation of an integral, 14.
Graphical representation of a definite integral, 73.
Gray, 189.
Gregory, 180.

Hele Shaw, 189.
Henrici's report on planimeter, 189.
Hermann, 188.
Hyperbola, area, 68.
  orthogonal trajectories, 217.
  related volumes, 77.
Hyperbolic spiral, area, 138.
  length, 149.
Hypocycloid, center of mass, 172, 173.
  intrinsic equation, 152.
  length, 147.
  surface, 156.
  volume of revolution, 76.

Integrable form, 37.
Integral, complete, 203.
Integral curves, 33, 190–200.
  applications, 195–198.
  applications to mechanics, 195, 240–242.
  applications to engineering and electricity, 242–244.
  determination of, 198.
  relations, analytical, 192.
  relations, geometrical, 194.
  relations, mechanical, 195.
Integral, definite, 8.
  definite, evaluation by measuring areas, 181.
  definite, geometrical representation, 14.
  definite, graphical representation, 73.
  definite, limits, 9.
  definite, precautions in finding, 67.
  definite, properties, 15.
  definite, relation to indefinite, 24.

Integral, elliptic, 147, 180.
  general, 23.
  indefinite, 22.
  indefinite, directions for finding, 54.
  multiple, 120.
  name, 1, 21.
  particular, 23, 204, 222.
Integrals, derivation of, 48.
  fundamental, 36–38, 47.
  table of, 249–262.
Integraph, 198, 200.
  theory of, 244.
Integrating factors, 207.
Integration, aided by changing variable, 41, 141.
  approximate, 177–189, 235–237
  by parts, 44, 46, 100, 231, 233.
  constants, 22, 202.
  definition, 1, 18, 21.
  derivation of reduction formulæ, 231, 233.
  fundamental formulæ, 37, 47.
  fundamental rules and methods, 36.
  in series, 177–179.
  mechanical, 188, 189, 200.
  of a total differential, 52.
  precautions, 63, 67.
  sign, 2, 21.
  signs in successive integration, 125.
  successive, one variable, 119–123.
  successive, two variables, 123–125.
  universal formulæ, 39, 40, 44.
  uses of, 1.
Intrinsic equation of a curve, 149.
Irrational functions, 84–99.

Jevons, 30.

Lamb, 237.
Laurent, 180.
Legendre, 180.
Leibniz, 2, 59, 144, 180.
Lemniscate, area, 137.
Lengths of curves, polar coördinates, 147.
  rectangular coördinates, 144.
Limits of a definite integral, 9.
Logarithmic curve, area, 76.
  length, 165.
Logarithmic spiral, area, 137.
  length, 149.

Markoff, 186.
Mass, 167.
Mass, center of, 168, 169.
Mean value, 17, 160–164.
  definition, 163.
Mechanical integration, 188–189, 200.
Multiple angles used, 114.
  integral, 120.

Neil, 144.
Newton, 2, 59, 73, 144, 180, 236.

Oliver, 4.
Orthogonal trajectories, 214–217.

Parabola, area, 5, 14, 59, 68, 76, 126, 138.
  center of mass, 173.
  derivation of equation, 26.
  intrinsic equation, 151, 152.
  length, 146, 149.
  length of evolute, 165.
  orthogonal trajectories, 216, 217.
  semicubical, area, 68.
  semicubical, intrinsic equation, 152.
  semicubical, length, 144, 147.
  surfaces of revolution, 155, 164.
  volume of revolution, 71, 77.
Parabolic rule, 184, 186, 236.
Paraboloid, center of mass, 173.
  volume, 131, 143.
Pascal, 59.
Planimeter described, 188, 189.
  theory of, 237.
Pyramid, volume, 143.

Quadrature, 58, 73.

Range, 161, 163.
Rational fractions, 78–83.
  decomposition of, 229–231.
Reciprocal substitution, 84.
Reduction formulæ, 46, 93–95, 101, 102, 106, 110, 231–233.
Roberval, 58.

Signs of integration, 2, 21, 125.
Simpson, Thomas, 184.
  one-third rule, 182, 184, 186.
  three-eighths rule, 236.
Slope of a curve, 25.
Solids of revolution, surfaces, 152–156.
  volumes, 69.
Solutions, general, 202.
Sphere, surface, 155, 158.
  volume, 131, 132, 133, 143, 164.

Spheroid, surface, 155.
  volume, 72.
Spiral of Archimedes, area, 138.
  length, 149.
Substitution, 41, 54.
  reciprocal, 84.
  trigonometric, 85.
Summation of infinitesimals, 1, 2, 234, 235.
Surfaces of revolution, areas, 152–156.
  volumes, 69.
  $f(x, y, z) = 0$, areas, 156–160.
  $f(x, y, z) = 0$, volumes, 126–133, 128.

Table of integrals, 249–262.
Torus, surface, 164.
  volume, 143.
Total differential, integration of, 52.
Tractrix, intrinsic equation, 152.
Trajectories, orthogonal, 214–217.
Transformation, algebraic, 103, 105, 107, 108, 113.
Trapezoidal rule, 182, 186, 236.
Trigonometric functions, integration of, 100–118.
  substitutions, 85.

Undetermined coefficients, use of, 93, 109.
Universal formulæ of integration, 39, 40, 44.
Uses of integral calculus, 1.

Volumes, of revolution, 69.
  polar coördinates, 131.
  rectangular coördinates in general, 128.

Wallis, 59, 144.
Weddle, 236.
Williamson, 180, 235.
Witch, area, 64.
  volume, 164.
Wren, 144.

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